Laurent Polynomials, GKZ-hypergeometric Systems and Mixed Hodge Modules

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Abstract

Given a family of Laurent polynomials, we will construct a morphism between its (proper) Gauß-Manin system and a direct sum of associated GKZ systems. The kernel and cokernel of this morphism are very simple and consist of free $\mathcal{O}$-modules. The result above enables us to put a mixed Hodge module structure on certain classes of GKZ systems and shows that they have quasi-unipotent monodromy.

Introduction

At the end of the 80's Gelfand, Kapranov and Zelevinsky introduced differential equations, which are a vast generalization of Gauß's hypergeometric equation and which are nowadays called GKZ-hypergeometric systems. Since then GKZ systems found applications in many fields of mathematics like representation theory, combinatorics and in particular in the area of mirror symmetry. There it was found that the periods of some families of Calabi-Yau varieties are among the solutions of certain resonant GKZ systems. In another direction Adolphson and Sperber [AS10] showed that a non-resonant, homogeneous GKZ system is isomorphic to a direct factor of a Gauß-Manin system of a family of affine varieties. In this paper we compare the Gauß-Manin system of a family of Laurent polynomials to GKZ systems which are not strongly resonant in the sense of [SW09]. A benefit of our approach is that we can endow these GKZ systems with the structure of a polarizable mixed Hodge module which enables us to prove that their (local) monodromies are quasi-unipotent.

Let us describe the results of this paper in some more detail. Given a family of Laurent polynomials

$$\varphi_A : S \times \mathbb{C}^n \longrightarrow \mathbb{C}_{\lambda_0} \times \mathbb{C}^n\,,$$
$$(y_1, \ldots, y_d, \lambda_1, \ldots, \lambda_n) \mapsto (-\sum_{i=1}^{n} \lambda_i \prod_{k=1}^{d} y_k^{a_{ki}}, \lambda_1, \ldots, \lambda_n)\,,$$

where $S = (\mathbb{C}^*)^d$ and A is a $d \times n$ matrix with integer entries (a_{ki}), we associate to it the matrix

$$\widetilde{A} := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}. \tag{0.0.1}$$

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The GKZ system $\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}}$ with respect to the matrix $\widetilde{A}$ and parameter vector $\widetilde{\beta} \in \mathbb{C}^{d+1}$ is a cyclic left $\mathcal{D}$-module on the affine space $V := \mathbb{C}_{\lambda_0} \times \mathbb{C}^n$.

Denote by $\mathbb{N}\widetilde{A}$ the semigroup generated by the columns of $\widetilde{A}$. Assume for the moment that this semigroup is saturated, i.e. $\mathbb{N}\widetilde{A} = \mathbb{R}_+\widetilde{A} \cap \mathbb{Z}^{d+1}$ and that $\mathbb{Z}\widetilde{A} = \mathbb{Z}^{d+1}$.

For each $\widetilde{\beta} \in \mathbb{N}\widetilde{A}$ we will construct a morphism from the Gauß-Manin system $\mathcal{H}^0(\varphi_{A,+}\mathcal{O}_{S\times\mathbb{C}^n})$ of the family of Laurent polynomials φ_A to this GKZ-system $\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}}$, the kernel and cokernel of which are free $\mathcal{O}$-modules and can be described explicitly. This gives rise to the following exact sequences

$$0 \longrightarrow H^{n-1}(S,\mathbb{C}) \otimes \mathcal{O}_{\mathbb{C}\times\mathbb{C}^n} \longrightarrow \mathcal{H}^0(\varphi_{A,+}\mathcal{O}_{S\times\mathbb{C}^n}) \longrightarrow \mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}} \longrightarrow H^n(S,\mathbb{C}) \otimes \mathcal{O}_{\mathbb{C}\times\mathbb{C}^n} \longrightarrow 0\,.$$

By holonomic duality we will get for each $\widetilde{\beta}$ lying in the interior of $\mathbb{N}\widetilde{A}$ a morphism from the GKZ system $\mathcal{M}_{\widetilde{A}}^{-\widetilde{\beta}}$ to the proper Gauß-Manin system of φ_A, which gives the following exact sequences

$$0 \longleftarrow H_{n-1}(S,\mathbb{C}) \otimes \mathcal{O}_{\mathbb{C}\times\mathbb{C}^n} \longleftarrow \mathcal{H}^0(\varphi_{A,\dagger}\mathcal{O}_{S\times\mathbb{C}^n}) \longleftarrow \mathcal{M}_{\widetilde{A}}^{-\widetilde{\beta}} \longleftarrow H_n(S,\mathbb{C}) \otimes \mathcal{O}_{\mathbb{C}\times\mathbb{C}^n} \longleftarrow 0\,.$$

It turns out that the two sequences above are global versions of the well-known long-exact sequences for the (co)homology of the pair $(S, \varphi^{-1}(\underline{\lambda}^0))$, i.e if we restrict them to a generic point $\underline{\lambda}^0 = (\lambda_0^0, \ldots, \lambda_n^0)$ we get

$$0 \longrightarrow H^{n-1}(S,\mathbb{C}) \longrightarrow H^{n-1}(\varphi^{-1}(\underline{\lambda}^0),\mathbb{C}) \longrightarrow H^n(S,\varphi^{-1}(\underline{\lambda}^0),\mathbb{C}) \longrightarrow H^n(S,\mathbb{C}) \longrightarrow 0\,,$$
$$0 \longleftarrow H_{n-1}(S,\mathbb{C}) \longleftarrow H_{n-1}(\varphi^{-1}(\underline{\lambda}^0),\mathbb{C}) \longleftarrow H_n(S,\varphi^{-1}(\underline{\lambda}^0),\mathbb{C}) \longleftarrow H_n(S,\mathbb{C}) \longleftarrow 0\,,$$

which are related by Poincaré duality.

Notice that the latter two exact sequences are not just exact sequences of vector spaces, but exact sequences in the category of mixed Hodge structures. A key feature of our proof is the fact that we can endow the GKZ systems above with a natural mixed Hodge module structure such that the first two sequences become exact sequences in the category of mixed Hodge modules.

The construction of the sequences above goes back to the work of [GKZ90] who studied GKZ systems by looking at their Fourier-Laplace transformation. They proved that for a non-resonant parameter $\widetilde{\beta}$ the Fourier-Laplace transformed GKZ system is isomorphic to the direct image of the twisted structure sheaf $\mathcal{O}_T\underline{y}^{\widetilde{\beta}}$ under the torus embedding

$$\begin{aligned} h : T \simeq (\mathbb{C}^*)^{d+1} &\longrightarrow \mathbb{C}^{n+1}\,, \\ (y_0,\ldots,y_d) &\mapsto (\prod_{k=0}^{d} y_k^{\widetilde{a}_{k0}}, \ldots, \prod_{k=0}^{d} y_k^{\widetilde{a}_{kn}})\,, \end{aligned}$$

This was further generalized by Schulze and Walther in [SW09] to parameter vectors $\widetilde{\beta}$ which are not in the set of strongly resonant parameters. It follows from a classification theorem of Saito [Sai01] that the set of integer $\widetilde{\beta}$ for which $h_+\mathcal{O}_T \simeq \mathrm{FL}\,\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}}$ holds is exactly the set $\mathbb{N}\widetilde{A}$ and the dual statement $h_\dagger\mathcal{O}_T \simeq \mathrm{FL}\,\mathcal{M}_{\widetilde{A}}^{-\widetilde{\beta}}$ holds exactly for $\widetilde{\beta}$ in the interior of $\mathbb{N}\widetilde{A}$.

In order to relate this to the Gauß-Manin system of φ_A, we first observe that the fibers of φ_A can be identified with hyperplane sections of a locally closed subvariety of $\mathbb{P}^n$. To be more precise, consider the torus embedding

$$\begin{aligned} g : S \simeq (\mathbb{C}^*)^d &\longrightarrow \mathbb{P}^n\,, \\ (y_1,\ldots,y_d) &\mapsto (\mu_0 : \ldots : \mu_n) = (1 : \prod_{k=1}^{d} y_k^{a_{k0}} : \ldots : \prod_{k=1}^{d} y_k^{a_{kn}})\,. \end{aligned}$$

The intersection of S with a hyperplane $H_{\underline{\lambda}^0}$, given by the equation $\sum_{i=0}^n \lambda_i^0 \mu_i = 0$, is the subvariety in S given by the equation

$$\lambda_0^0 + \sum_{i=1}^n \lambda_0^i \prod_{k=1}^n y_k^{a_{ki}} = 0\,,$$

which is easily seen to be equal to the fiber of φ_A at $\underline{\lambda}^0$.

The Radon transform of a $\mathcal{D}$-module on $\mathbb{P}^n$ was defined in [Bry86]. Loosely speaking it measures the restriction of the $\mathcal{D}$-module to a hyperplane of $\mathbb{P}^n$ parametrized by $\mathbb{C}^{n+1}$. If one applies this transformation to the $\mathcal{D}$-module $g_+\mathcal{O}_S$, the discussion above shows that its Radon transform is isomorphic to the Gauß-Manin system of φ_A.

We use a comparison theorem of d'Agnolo and Eastwood [DE03] between the Radon transformation and Fourier-Laplace tansformation. Given a $\mathcal{D}$-module $\mathcal{N}$ on $\mathbb{P}^n$, its Radon transform is isomorphic to the Fourier-Laplace transform of a certain extension from $\mathbb{C}^{n+1} \setminus \{0\}$ to $\mathbb{C}^{n+1}$ of the inverse image of $\mathcal{N}$ under the quotient map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Applying this to the $\mathcal{D}$-module $g_+\mathcal{O}_S$ shows that the that the extension of $\pi^+ g_+\mathcal{O}_S$ to $\mathbb{C}^{n+1}$ differs from $h_+\mathcal{O}_T$ only at $0 \in \mathbb{C}^{n+1}$. But this shows that after Fourier-Laplace transformation the Gauß-Manin system of φ_A and the GKZ-system $\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}}$ differ for $\widetilde{\beta} \in \mathbb{N}\widetilde{A}$ only by free $\mathcal{O}_V$-modules.

If one drops the assumption that $\mathbb{N}\widetilde{A}$ is a saturated semigroup the situation becomes more complicated. Given a family of Laurent polynomials φ_B, we only assume that the columns of B generate $\mathbb{Q}^d$ over $\mathbb{Q}$, i.e. $\mathbb{Q}B = \mathbb{Q}^d$. Denote by $e = (e_1, \ldots, e_d) \in \mathbb{N}^d$ the elementary divisors of B and let A be the matrix constructed from B by setting all elementary divisors equal to 1. In this case we get again an morphism

$$\mathcal{H}^0(\varphi_{B,+}\mathcal{O}_S) \longrightarrow \bigoplus_{\gamma \in I_e} \mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}+(0,\gamma)}$$

with constant kernel and cokernel, where the set I_e depends on the elementary divisors of B.

A key feature of the presentation of a GKZ system as a Radon transformation is the fact that any regular holonomic GKZ system with an integer parameter vector, which is not strongly resonant, carries a polarizable mixed Hodge module structure. This opens the door to a Hodge theoretic study of the GKZ systems. As an easy application we prove that the (local) monodromies of these systems are quasi-unipotent.

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1 Preliminaries

In the first two sections we will collect, for the reader's convenience, some facts about $\mathcal{D}$-modules and mixed Hodge modules. In section 1.3 we review the Radon and Fourier-Laplace transformations for (monodromic) $\mathcal{D}$-modules and state a comparison theorem between these transformations due to [DE03]. In section 1.4 we will give the definition of a GKZ system and state a theorem of Schulze and Walther [SW09] which expresses the Fourier-Laplace transformation of such a system as a direct image of a torus-embedding, when the parameter vector β is not in the set of so-called strongly resonant values. We will prove some basic facts of this set which will be essential in the following. We will also review a theorem of Walther [Wal07] on the holonomic dual of a GKZ-system and state some results on isomorphism classes of GKZ systems due to Saito [Sai01].

1.1 $\mathcal{D}$-modules

In this section X is a smooth algebraic variety over $\mathbb{C}$ of dimension n. We denote by $M(\mathcal{D}_X)$ the abelian category of algebraic $\mathcal{D}_X$-modules on X and the abelian subcategory of (regular) holonomic $\mathcal{D}_X$-modules by $M_h(\mathcal{D}_X)$ (resp. $(M_{rh}(\mathcal{D}_X))$. The full triangulated subcategory in $D^b(\mathcal{D}_X)$, consisting of objects with (regular) holonomic cohomology, is denoted by $D^b_h(\mathcal{D}_X)$ (resp. $D^b_{rh}(\mathcal{D}_X)$).

Let $f : X \to Y$ be a map between smooth algebraic varieties. Let $M \in D^b(\mathcal{D}_X)$ and $N \in D^b(\mathcal{D}_Y)$, then we denote by $f_+M := Rf_*(\mathcal{D}_{Y\leftarrow X} \overset{L}{\otimes} M)$ resp. $f^\dagger M := \mathcal{D}_{X\to Y} \overset{L}{\otimes} f^{-1}(M)$ the direct resp. inverse image for $\mathcal{D}$-modules. Notice that the functors $f_+, f^\dagger$ preserve (regular) holonomicity (see e.g., [HTT08, Theorem 3.2.3]). We denote by $\mathbb{D} : D^b_h(\mathcal{D}_X) \to (D^b_h(\mathcal{D}_X))^{opp}$ the holonomic duality functor. Recall that for a single holonomic $\mathcal{D}_X$-module M, the holonomic dual is also a single holonomic $\mathcal{D}_X$-module ([HTT08, Proposition 3.2.1]) and that holonomic duality preserves regular holonomicity ([HTT08, Theorem 6.1.10]).

For a morphism $f : X \to Y$ between smooth algebraic varieties we additionally define the functors $f_\dagger := \mathbb{D} \circ f_+ \circ \mathbb{D}$ and $f^+ := \mathbb{D} \circ f^\dagger \circ \mathbb{D}$.

Let $i : Z \to X$ be a closed embedding of a smooth subvariety of codimension d and $j : U \to X$ be the open embedding of its complement. This gives rise to the following triangles for $M \in D^b_{rh}(\mathcal{D}_X)$

$$i_+i^\dagger M[-d] \longrightarrow M \longrightarrow j_+j^\dagger M \xrightarrow{+1}, \tag{1.1.1}$$

$$j_\dagger j^+ M \longrightarrow M \longrightarrow i_+i^+M[d] \xrightarrow{+1} . \tag{1.1.2}$$

The first triangle is [HTT08, Proposition 1.7.1] and the second triangle follows by dualization, where we have used that $i_+ = i_\dagger$.

We will often use the following base change theorem.

Theorem 1.1 (Theorem 1.7.3). *[HTT08] Consider the following cartesian diagram of algebraic varieties*

$$\begin{tikzcd} Z \arrow[r, "f'"] \arrow[d, "g'"'] & W \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

then we have the canonical isomorphism $f^\dagger g_+[d] \simeq g'_+ f'^\dagger[d']$, *where* $d := \dim Y - \dim X$ *and* $d' := \dim Z - \dim W$.

Remark 1.2. *Notice that by symmetry we have also the canonical isomorphism* $g^\dagger f_+[\tilde{d}] \simeq f'_+ g'^\dagger[\tilde{d}']$ *with* $\tilde{d} := \dim W - \dim X$ *and* $\tilde{d}' := \dim Z - \dim Y$. *In the former case we say we are doing a base change with respect to* f, *in the latter case with respect to* g.

Remark 1.3. *Using the duality functor we get isomorphisms:*

$$f^+g_\dagger[-d] \simeq g'_\dagger {f'}^+[-d'] \qquad \textit{and} \qquad g^+f_\dagger[-\tilde{d}] \simeq f'_\dagger {g'}^+[-\tilde{d}'] \,.$$

Remark 1.4. *If f and f' are smooth maps with the same fiber dimension l, one can easily show that $f^+g_+ \simeq g'_+{f'}^+$ by using $f^\dagger = f^+$ resp. ${f'}^\dagger = {f'}^+$ (c.f.* [BGK+87] *VI, Corollary 9.14). (Notice that* [BGK+87] *uses shifted version for $f^\dagger$ resp. f^+.)*

1.2 Mixed Hodge Modules

In this section we introduce mixed Hodge modules and corresponding functors together with some of their properties through an axiomatic approach in the spirit of [PS08].

Let X be a smooth algebraic variety. We denote by $D^b_c(X,\mathbb{Q})$ the derived category of bounded, algebraic, constructible sheaves of $\mathbb{Q}$-vector spaces and by $Perv(X,\mathbb{Q})$ the heart of its perverse t-structure. There exists an abelian category $MHM(X)$ together with faithful functors $rat : D^bMHM(X) \to D^b_c(X,\mathbb{Q})$ such that $MHM(X)$ corresponds to $Perv(X,\mathbb{Q})$ and $Dmod : D^bMHM(X) \to D^b_{rh}(X)$ such that for any complex $M \in D^bMHM(X)$ we have the following comparison isomorphism

$$(rat \otimes \mathbb{C})\,(M) \stackrel{\simeq}{\to} DR_X(Dmod(M))\,.$$

The duality functor $\mathbb{D}$ lifts to $D^bMHM(X)$, also denoted by $\mathbb{D}$, in the sense that $rat \circ \mathbb{D} \simeq \mathbb{D} \circ rat$ resp. $\mathbb{D} \circ Dmod \simeq Dmod \circ \mathbb{D}$.
For each morphism $f : X \to Y$ between complex algebraic varieties, there are induced functors $f_*, f_! : D^bMHM(X) \to D^bMHM(Y)$, $f^*, f^! : D^bMHM(Y) \to D^bMHM(X)$ which are interchanged by $\mathbb{D}$ and which lift the analogous functors $f_+, f_\dagger, f^+[-d], f^\dagger[d]$ on $D^b_{rh}(\mathcal{D}_X)$ resp. $Rf_*, f_!, f^{-1}, f^!$ on $D^b_c(X)$, where $d := \dim X - \dim Y$.

Recall that by [Sai90] (4.4.3) the base change theorem above holds also in the category of algebraic mixed Hodge modules.

Let $\mathbb{Q}^H_{pt}$ be the unique mixed Hodge structure of weight 0 with $Gr^W_i = 0$ for $i \neq 0$ and underlying vectorspace $\mathbb{Q}$. Denote by $a_X : X \longrightarrow \{pt\}$ the map to the point. We define $\mathbb{Q}^H_X := a^*_X\mathbb{Q}^H_{pt}$.

1.3 Radon and Fourier-Laplace Transformations

For the reader's convenience, we recall in this subsection some definitions and general facts about Fourier-Laplace and Radon transformations for $\mathcal{D}$-modules. Let X be a smooth algebraic variety, V be a finite-dimensional complex vector space and V' its dual vector space. Denote by E' the trivial vector bundle $\tau : X \times V' \to X$ and by E its dual, with fiber coordinates $\underline{\mu} = (\mu_0, \dots, \mu_n)$ resp. $\underline{\lambda} = (\lambda_0, \dots, \lambda_n)$. Let $\langle\cdot,\cdot\rangle$ be the standard euclidean pairing with respect to these coordinates.

Definition 1.5. *Define $\mathcal{L} := \mathcal{D}_{E'\times_X E}e^{-\langle\underline{\mu},\underline{\lambda}\rangle}$ and denote by $p_1 : E'\times_X E \to E'$, $p_2 : E'\times_X E \to E$ the canonical projections. The Fourier-Laplace transformation is then defined by*

$$\mathrm{FL}_X(M) := p_{2+}(p_1^+M \overset{L}{\otimes} \mathcal{L}) \quad M \in D^b_h(\mathcal{D}'_E).$$

If X is a point we will simply write FL. In general, the Fourier-Laplace transformation does not preserve regular holonomicity. But for the derived category of complexes of $\mathcal{D}$-modules with monodromic cohomology, regular holonomicity is preserved. Let $\theta : \mathbb{C}^* \times E' \to E'$ be the natural $\mathbb{C}^*$ action on the fiber V' and let z be a coordinate on $\mathbb{C}^*$. We denote the push-forward $\theta_*(z\partial_z)$ as the Euler vector field $\mathfrak{E}$.

Definition 1.6. *[Bry86] A regular holonomic $\mathcal{D}_{E'}$-module M is called monodromoic, if the Euler field $\mathfrak{E}$ acts locally finite on $\tau_*(M)$, i.e. for a local section v of $\tau_*(M)$ the set $\mathfrak{E}^n(v)$, $(n \in \mathbb{N})$, generates a finite-dimensional vector space. We denote by $D^b_{mon}(\mathcal{D}_{E'})$ the derived category of bounded complexes of $\mathcal{D}_{E'}$-modules with regular holonomic and monodromic cohomology.*

Theorem 1.7. *[Bry86]*

1. FL *preserves complexes with monodromic cohomology.*
2. *In $D^b_{mon}(\mathcal{D}_{E'})$ we have*
$$\mathrm{FL} \circ \mathrm{FL} \simeq Id \quad \textit{and} \quad \mathbb{D} \circ \mathrm{FL} \simeq \mathrm{FL} \circ \mathbb{D}\,.$$
3. FL *is t-exact with respect to the natural t-structure on $D^b_{mon}(\mathcal{D}_{E'})$ resp. $D^b_{mon}(\mathcal{D}_E)$.*

Proof. The above statements are stated in [Bry86] for constructible monodromic complexes. One has to use the Riemann-Hilbert correspondence, [Bry86, Proposition 7.12, Theorem 7.24] to translate the statements. So the first statement is Corollaire 6.12, the second statement is Proposition 6.13 and the third is Corollaire 7.23 in [Bry86]. □

Remark 1.8. *By Proposition 7.12 of [Bry86] a complex in $D^b_{mon}(\mathcal{D}_{E'})$ corresponds to a complex of sheaves of $\mathbb{C}$-vector spaces with constructible, monodromic cohomology, where we call a sheaf of $\mathbb{C}$-vector spaces monodromic if it is locally constant along the orbits of the $\mathbb{C}^*$-action on the fibers of E'. Let $\pi_{E'} : E' \setminus 0_{E'} \to \mathbb{P}(E')$ be the projectivization of E', where $0_{E'}$ is the zero section of E'. This characterization shows that a complex $M \in D^b_{mon}(\mathcal{D}_{E'})$ is monodromic if its restriction to the complement of the zero section is isomorphic to $\pi^+_{E'} N$ for some $N \in D^b_{rh}(\mathcal{D}_{\mathbb{P}(E')})$.*

The second main tool that we will use later are variants of the so-called Radon transform. On the level of $\mathcal{D}$-modules the Radon transform was discussed by [Bry86] and its variants were later discussed by [DE03].

Consider the following diagram

$$\begin{tikzcd}
 & U \arrow[dl, "\pi_1^U"'] \arrow[d, hook, "j_U"] \arrow[dr, "\pi_2^U"] & \\
\mathbb{P}(V') & \mathbb{P}(V') \times V \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & V\,, \\
 & Z \arrow[ul, "\pi_1^Z"] \arrow[u, hook', "i_Z"'] \arrow[ur, "\pi_2^Z"'] &
\end{tikzcd}$$

where we denote by Z the hypersurface given by the equation $\sum_{i=0}^n \mu_i \lambda_i = 0$ and by U its complement in $\mathbb{P}(V') \times V$. The Radon transformation is a functor $\mathcal{R} : D^b_{rh}(\mathcal{D}_{\mathbb{P}(V')}) \longrightarrow D^b_{rh}(\mathcal{D}_V)$ given by
$$\mathcal{R}(M) := \pi^Z_{2+}\,(\pi^Z_1)^+ M \simeq \pi_{2+}\, i_{Z+}\, i^+_Z\, \pi^+_1 M\,,$$
its variant $\mathcal{R}^\circ : D^b_{rh}(\mathcal{D}_{\mathbb{P}(V')}) \longrightarrow D^b_{rh}(\mathcal{D}_V)$ is defined by
$$\mathcal{R}^\circ(M) := \pi^U_{2+}\,(\pi^U_1)^+ M \simeq \pi_{2+}\, j_{U+} j^+_U \pi^+_1 M\,,$$
together with a compact version $\mathcal{R}^\circ_c : D^b_{rh}(\mathcal{D}_{\mathbb{P}(V')}) \longrightarrow D^b_{rh}(\mathcal{D}_V)$ which is defined by
$$\mathcal{R}^\circ_c(M) := \pi^U_{2\dagger}\,(\pi^U_1)^+ M \simeq \pi_{2+}\, j_{U\dagger}\, j^+_U \pi^+_1 M\,.$$
Finally we introduce the transformation $\mathcal{R}_{cst} : D^b_{rh}(\mathcal{D}_{\mathbb{P}(V')}) \longrightarrow D^b_{rh}(\mathcal{D}_V)$ which is defined by
$$\mathcal{R}_{cst}(M) := \pi_{2+}\,(\pi_1)^+ M\,.$$
Notice that the various Radon transformations give rise to the following triangles:

Proposition 1.9. *Let $M \in D^b_{rh}(\mathcal{D}_{\mathbb{P}(V')})$, we have*

$$\mathcal{R}(M)[-1] \longrightarrow \mathcal{R}_{cst}(M) \longrightarrow \mathcal{R}^\circ(M) \xrightarrow{+1},$$
$$\mathcal{R}^\circ_c(M) \longrightarrow \mathcal{R}_{cst}(M) \longrightarrow \mathcal{R}(M)[1] \xrightarrow{+1},$$

where the second triangle is dual to the first.

Proof. The first triangle follows from the adjunction triangle (1.1.1) and the second from the triangle (1.1.2). ☐

The following proposition relates the Fourier and Radon transformations introduced above, and will be quite useful in the next chapter.

Proposition 1.10. *[DE03, Proposition 1] Let V be a $\mathbb{C}$-vector space, V' its dual space, $p : Bl_0(V') := \mathbb{V}(\mathcal{O}_{\mathbb{P}(V')}(-1)) \to V'$ the blowup of V' at the origin, and consider the following diagram*

$$\begin{tikzcd}
 & V' \arrow[dr, dotted, "\mathrm{FL}"] & \\
Bl_0(V') \arrow[ur, "p"] \arrow[dr, "q"'] & V' \setminus \{0\} \arrow[l] \arrow[u, "j"] \arrow[d, "\pi"'] & V\,. \\
 & \mathbb{P}(V') \arrow[ur, dotted, bend left, "\mathcal{R}^\circ_{(c)}"] \arrow[ur, dotted, bend right, "\mathcal{R}_{(cst)}"'] &
\end{tikzcd}$$

Let $M \in D^b_{rh}(\mathcal{D}_{\mathbb{P}(V')})$. Then we have

$$\mathcal{R}(M) \simeq \mathrm{FL}(p_+q^+M),$$
$$\mathcal{R}^\circ_c(M) \simeq \mathrm{FL}(j_+\pi^+M),$$
$$\mathcal{R}^\circ(M) \simeq \mathrm{FL}(j_\dagger\pi^+M).$$

We can define the various Radon transforms also in the derived category of mixed Hodge modules. Define ${}^*\mathcal{R} : D^bMHM(\mathbb{P}(V')) \to D^bMHM(V)$ by

$${}^*\mathcal{R}(M) := \pi^Z_{2*}(\pi^Z_1)^*M \simeq \pi_{2*}i_{Z*}i^*_Z\pi^*_1M$$

and ${}^*\mathcal{R}_{cst} : D^bMHM(\mathbb{P}(V')) \to D^bMHM(V)$

$${}^*\mathcal{R}_{cst}(M) := \pi_{2*}\pi^*_1M\,.$$

Notice that unlike in the category of $\mathcal{D}$-modules ${}^*\mathcal{R}$ and ${}^*\mathcal{R}_{cst}$ commute with the duality functor $\mathbb{D}$ only up to shift and Tate twist. We therefore define ${}^!\mathcal{R} : D^bMHM(\mathbb{P}(V')) \longrightarrow D^bMHM(V)$

$${}^!\mathcal{R}(M) := \mathbb{D} \circ {}^*\mathcal{R} \circ \mathbb{D}\,(M) \simeq \pi^Z_{2*}(\pi^Z_1)^!M \simeq \pi_{2*}i_{Z*}i^!_Z\pi^!_1M$$

and ${}^!\mathcal{R}_{cst} : D^bMHM(\mathbb{P}(V')) \longrightarrow D^bMHM(V)$

$${}^!\mathcal{R}_{cst}(M) := \mathbb{D} \circ {}^*\mathcal{R}_{cst} \circ \mathbb{D}\,(M) \simeq \pi_{2*}\pi^!_1M\,.$$

Finally we define ${}^*\mathcal{R}^\circ_c : D^bMHM(\mathbb{P}(V')) \to D^bMHM(V)$ by

$${}^*\mathcal{R}^\circ_c(M) := \pi^U_{2!}(\pi^U_1)^*(M) \simeq \pi_{2*}j_{U!}j^*_U\pi^*_1(M)$$

and ${}^!\mathcal{R}^\circ : D^bMHM(\mathbb{P}(V') \to D^bMHM(V)$ by

$${}^!\mathcal{R}^\circ(M) := \mathbb{D} \circ {}^*\mathcal{R}^\circ_c \circ \mathbb{D}\,(M) \simeq \pi^U_{2*}(\pi^U_1)^!(M) \simeq \pi_{2*}j_{U*}j^!_U\pi^!_1(M)\,.$$

Using these definitions we get the triangles equivalent to Proposition 1.9.

Proposition 1.11. *Let $M \in D^b MHM(\mathbb{P}(V'))$, we have the following triangles*

$$
{}^!\mathcal{R}(M) \longrightarrow {}^!\mathcal{R}_{cst}(M) \longrightarrow {}^!\mathcal{R}^\circ(M) \xrightarrow{+1},
$$
$$
{}^*\mathcal{R}_c^\circ(M) \longrightarrow {}^*\mathcal{R}_{cst}(M) \longrightarrow {}^*\mathcal{R}(M) \xrightarrow{+1},
$$

where the second triangle is dual to the first.

Proof. The proof is the same as in Proposition 1.9 using [Sai90, (4.4.1)]. □

1.4 GKZ Systems

Definition 1.12 ([GKZ90], [Ado94])**.** *Consider a lattice $\mathbb{Z}^d$ and vectors $\underline{a}_1, \ldots, \underline{a}_n \in \mathbb{Z}^d$ which we also write as a matrix $A = (\underline{a}_1, \ldots, \underline{a}_n)$. In the following we assume that the vectors $\underline{a}_1, \ldots, \underline{a}_n$ generate $\mathbb{Z}^d$ as a $\mathbb{Z}$-module. Moreover, let $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{C}^d$. Write $\mathbb{L}$ for the module of relations among the columns of A and $\mathcal{D}_{\mathbb{C}^n}$ for the sheaf of rings of algebraic differential operators on $\mathbb{C}^n$ (where we choose $\lambda_1, \ldots, \lambda_n$ as coordinates). Define*

$$
\mathcal{M}_A^\beta := \mathcal{D}_{\mathbb{C}^n} / \left((\Box_{\underline{l}})_{\underline{l} \in \mathbb{L}} + (E_k - \beta_k)_{k=1,\ldots d} \right),
$$

where

$$
\begin{aligned}
\Box_{\underline{l}} &:= \textstyle\prod_{i:l_i<0} \partial_{\lambda_i}^{-l_i} - \prod_{i:l_i>0} \partial_{\lambda_i}^{l_i}, \\
E_k &:= \textstyle\sum_{i=1}^n a_{ki} \lambda_i \partial_{\lambda_i}.
\end{aligned}
$$

$\mathcal{M}_A^\beta$ is called a GKZ system. It is a holonomic $\mathcal{D}$-module by [Ado94, Theorem 3.9].

As GKZ systems are defined on the affine space $\mathbb{C}^n$, we will often work with the D-modules of global sections $M_A^\beta := \Gamma(\mathbb{C}^n, \mathcal{M}_A^\beta)$ rather than with the sheaves themselves, where D is the Weyl Algebra $\mathbb{C}[\lambda_1, \ldots, \lambda_n]\langle \partial_{\lambda_1}, \ldots, \partial_{\lambda_n} \rangle$.

1.4.1 Fourier-Laplace transformed GKZ-systems

Denote by

$$
\mathbb{N}A := \{ \sum_{i=1}^d \gamma_i \underline{a}_i \in \mathbb{Z}^d \mid (\gamma_i)_{i=1,\ldots,d} \in \mathbb{N}^d \}
$$

the semigroup built by the columns of A seen as elements in $\mathbb{Z}^d$. The semigroup ring associated to the matrix A is $S_A := \mathbb{C}[\mathbb{N}A] \simeq R/I_A$, where R is the commutative ring $\mathbb{C}[\partial_{\lambda_1}, \ldots, \partial_{\lambda_n}]$, I_A is the ideal

$$
I_A = \{ \Box_{\underline{l}}, l \in \mathbb{L} \}.
$$

and the isomorphism follows from [MS05, Theorem 7.3]. The rings R and S_A are naturally $\mathbb{Z}^d$-graded if we define $\deg(\partial_{\lambda_j}) = -\underline{a}_j$ for $j = 1, \ldots, n$. This is compatible with the $\mathbb{Z}^d$-grading of the Weyl algebra D given by $\deg(\lambda_j) = \underline{a}_j$ and $\deg(\partial_{\lambda_j}) = -\underline{a}_j$.
If we do the substitution $\partial_{\lambda_i} \mapsto \mu_i$, the ideal I_A goes over to an ideal $\hat{I}_A \subset \mathbb{C}[\mu_1, \ldots, \mu_n]$. By definition of $\mathbb{L}$ as the module of relations of A, we have the following isomorphism

$$
\mathbb{C}[\mathbb{N}A] \simeq \mathbb{C}[\mu_1, \ldots, \mu_n]/\hat{I}_A .
$$

Furthermore, we have a ring homomorphism

$$
\begin{aligned}
\mathbb{C}[\mu_1, \ldots, \mu_n]/\hat{I}_A &\longrightarrow \mathbb{C}[y_1^{\pm}, \ldots, y_d^{\pm}], \\
\mu_i &\mapsto \underline{y}^{\underline{a}_i} := \prod_{k=1}^d y_k^{a_{ki}}.
\end{aligned}
$$

If we define $Y' := \operatorname{Spec}(\mathbb{C}[\mathbb{N}A])$ and $T' := \operatorname{Spec}(\mathbb{C}[y_1^{\pm}, \ldots, y_d^{\pm}])$, this gives rise to the following maps

$$T' \xrightarrow{k} Y' \xrightarrow{l} \mathbb{C}^n\,, \tag{1.4.1}$$

where k is an open and l a closed embedding. The fact that k is an open embedding follows from the assumption that the $\underline{a}_i$ generate $\mathbb{Z}^d$ and the definition of $\hat{I}_A$, which gives the following isomorphism, obtained from the ring homomorphism above:

$$\mathbb{C}[\mu_1^{\pm}, \ldots, \mu_n^{\pm}]/\hat{I}_A \simeq \mathbb{C}[y_1^{\pm}, \ldots, y_d^{\pm}]\,.$$

Because of the discussion above the hypergeometric system $\mathcal{M}_A^\beta$ can be seen as the total Fourier-Laplace transform of a $\mathcal{D}$-module which has support on Y'. In [SW09] it it shown that $\mathrm{FL}(\mathcal{M}_A^\beta)$ is isomorphic to $(l \circ k)_+ \mathcal{O}_{T'} \underline{y}^\beta$ for certain $\beta \in \mathbb{C}^d$, where we denote by $\mathcal{O}_{T'}\underline{y}^\beta$ the $\mathcal{D}$-module

$$\mathcal{D}_{T'}/\mathcal{D}_{T'} \cdot (y_1\partial_{y_1} - \beta_1, \ldots, y_d\partial_{y_d} - \beta_d)\,.$$

Definition 1.13 ([MMW05] Definition 5.2)**.** *Let N be a finitely generated $\mathbb{Z}^d$-graded R-module. An element $\alpha \in \mathbb{Z}^d$ is called a true degree of N if N_α is non-zero. A vector $\alpha \in \mathbb{C}^d$ is called a quasi-degree of N, written $\alpha \in qdeg(N)$, if α lies in the complex Zariski closure $qdeg(N)$ of the true degrees of N via the natural embedding $\mathbb{Z}^d \hookrightarrow \mathbb{C}^d$.*

Schulze and Walther now define the following set of parameters:

Definition 1.14 ([SW09])**.** *The set*

$$sRes(A) := \bigcup_{j=1}^{n} sRes_j(A)$$

is called the set of strongly resonant parameters of A, where

$$sRes_j(A) := \{\beta \in \mathbb{C}^d \mid \beta \in -(\mathbb{N}+1)\underline{a}_j - qdeg(S_A/\langle\partial_{\lambda_j}\rangle)\}.$$

The condition on β for $\mathrm{FL}(\mathcal{M}_A^\beta)$ being isomorphic to $(l \circ k)_+ \mathcal{O}_{T'}\underline{y}^\beta$ is now the following:

Theorem 1.15 ([SW09] Theorem 3.6, Corollary 3.7)**.** *Let $\mathbb{N}A$ be a positive semigroup, meaning that 0 is the only unit in $\mathbb{N}A$. Then the following is equivalent*

1. *$\beta \notin sRes(A)$.*
2. *$\mathcal{M}_A^\beta \simeq \mathrm{FL}((l \circ k)_+\, \mathcal{O}_{T'}\underline{y}^\beta)$.*
3. *Left multiplication with ∂_{λ_i} is invertible on M_A^β for $i = 1, \ldots, n$.*

In the next step we want to characterize the set $sRes(A)$. For this we have to understand the sets $qdeg(S_A/\langle\partial_{\lambda_j}\rangle)$ for $j = 1, \ldots, n$. We use the following definitions and notations of [MMW05]. A face of A is a set of columns of A minimizing some linear functional on $\mathbb{N}A \subset \mathbb{Z}^d$. Let S_F be the semigroup ring generated by F, then a $\mathbb{Z}^d$-graded R-module M is called toric if it has a toric filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_{l-1} \subset M_l = M\,,$$

meaning that, for each k, M_k/M_{k-1} is a $\mathbb{Z}^d$-graded translate of S_{F_k} for some face F_k of $\mathbb{N}A$, generated in degree $-\alpha_k$, which will be denoted by $S_{F_k}(\alpha_k)$.
Now by example 4.7 in [MMW05] one easily deduces that the $\mathbb{Z}^d$-graded rings $S_A/\langle\partial_j\rangle$ are toric for $j = 1, \ldots, n$.

The toric filtration

$$0 = M_0^j \subset M_1^j \subset \ldots \subset M_{l_j}^j = S_A/\langle\partial_{\lambda_j}\rangle \tag{1.4.2}$$

with $M^j_k/M^j_{k-1} \simeq S_{F_{k,j}}(\alpha_{k,j})$ gives us the following decomposition of the degrees

$$deg(S_A/\langle\partial_{\lambda_j}\rangle) = \bigcup_{k=1}^{l_j} deg(S_{F_{k,j}}(\alpha_{k,j}))$$

resp. of the quasidegrees

$$qdeg(S_A/\langle\partial_{\lambda_j}\rangle) = \bigcup_{k=1}^{l_j} qdeg(S_{F_{k,j}}(\alpha_{k,j})) \tag{1.4.3}$$

of $S_A/\langle\partial_{\lambda_j}\rangle$, where only faces $S_{F_{k,j}}$ occur which do not contain $\underline{a}_j$. The quasi-degrees of an $\mathbb{Z}^d$-graded R-module S_F, where F is a proper face of $\mathbb{N}A$, are just the $\mathbb{C}$-linear span of F, which we denote by $\mathbb{C}F$. (Here we use the embedding $\mathbb{Z}^d \hookrightarrow \mathbb{C}^d$). Thus the quasi-degrees of $S_A/\langle\partial_{\lambda_j}\rangle$ are just finite unions of translates of the linear subspaces $\mathbb{C}F_{k,j}$.

The following lemma shows that a translate of the cone $\mathbb{R}_+A$ does not meet the set of strongly resonant parameters. (Notice that the following lemma with $\delta_A = 0$ is stated in Corollary 3.8 of [SW09], but not proven. An easy counterexample of their claim is provided by $A = (2,5)$.)

Lemma 1.16. *Denote as above by $sRes(A)$ the set of strongly resonant vectors β. There exists $\delta_A \in \mathbb{N}A$ such that*

$$(\mathbb{R}_+A + \delta_A) \ \cap \ sRes(A) = \emptyset$$

Proof. Choose some non-zero vector α' in the interior of $\mathbb{N}A$, i.e. in $\mathbb{N}A \cap (\mathbb{R}_+A)^\circ$. As we assumed that $\mathbb{N}A$ is a positive semigroup, the set $\mathbb{R}_+A$ is a strongly convex cone in $\mathbb{R}^d$. We noticed above that $qdeg(S_F)$ is the $\mathbb{C}$-linear span of the face F, therefore we can conclude that

$$\mathbb{R}_+A + \alpha' + \alpha_{k,j} \cap -qdeg(S_{F_{k,j}}(\alpha_{k,j})) = \emptyset$$

where we have used $-qdeg(S_{F_{k,j}}(\alpha_{k,j})) = -qdeg(S_{F_{k,j}}) + \alpha_{k,j}$. As every $\alpha_{j,k}$ for $j = 1,\ldots,d$ and $k = 1,\ldots,l_j$ lies in $\mathbb{N}A$, the element $\alpha'' := \sum_{j=1}^d \sum_{k=1}^{l_j} \alpha_{k,j}$ also lies in $\mathbb{N}A$. Now set $\delta_A := \alpha' + \alpha''$. We have $\mathbb{R}_+A + \delta_A \subset \mathbb{R}_+A + \alpha' + \alpha_{k,j}$ and therefore

$$\mathbb{R}_+A + \delta_A \cap -qdeg(S_A/\langle\partial_{\lambda_j}\rangle) = \emptyset\,.$$

As $-a_j$ lies in $-\mathbb{N}A$, this shows that

$$\mathbb{R}_+A + \delta_A \cap -qdeg(S_A/\langle\partial_{\lambda_j}\rangle) - (l+1)a_j = \emptyset$$

for every $l \in \mathbb{N}$ and therefore $\mathbb{R}_+A + \delta_A \cap sRes_j(A) = \emptyset$. As this is true for every $j = 1,\ldots,d$, this shows the claim. ∎

The lemma above can be improved in an important special case. We call the semigroup $\mathbb{N}A$ saturated if

$$\mathbb{N}A = \mathbb{Q}_+A \cap \mathbb{Z}^d\,.$$

Lemma 1.17. *Let $\mathbb{N}A$ be a saturated semigroup then*

$$\mathbb{R}_+A \cap sRes(A) = \emptyset\,.$$

Proof. First notice that because $\mathbb{N}A$ is saturated the true degrees of S_A form exactly the set $-\mathbb{N}A$. Now observe that a monomial P in S_A is non-zero in $S_A/\langle\partial_{\lambda_j}\rangle$ if and only if $degP + a_j$ is not in $-\mathbb{N}A$.
As we observed above $qdeg(S_F)$ is the $\mathbb{C}$-linear span of a face F of $\mathbb{N}A$ (and with that also the $\mathbb{C}$-linear span of the face $-F$ of $-\mathbb{N}A$). If $a_j \notin F$ consider the finite set

$$I_{F,j} := \{t \in [0,1) \mid (qdeg(S_F) - t \cdot a_j) \cap (-\mathbb{N}A) \neq \emptyset\}\,.$$

The set

$$V_j := \bigcup_{F: a_j \notin F} \bigcup_{t \in I_{F,j}} qdeg(S_F) - t \cdot a_j = \bigcup_{F: a_j \notin F} \bigcup_{t \in I_{F,j}} \mathbb{C}F - t \cdot a_j$$

is Zariski-closed and contains the degrees of $S_A/\langle \partial_{\lambda_j} \rangle$ and therefore $qdeg(S_A/\langle \partial_{\lambda_j} \rangle) \subset V_j$. Looking now at the definition of $sRes_j(A)$ we see that

$$sRes_j(A) = \{\beta \mid \beta \in -(\mathbb{N}+1)a_j - qdeg(S_A/\langle \partial_{\lambda_j} \rangle)\} \; \subset \; \bigcup_{l \in \mathbb{N}} -(l+1)a_j - V_j \, .$$

But we clearly have $\mathbb{R}_+ A \; \cap \; \bigcup_{l \in \mathbb{N}} (-(l+1)a_j - V_j) = \emptyset$ and therefore $\mathbb{R}_+ A \cap sRes_j(A) = \emptyset$. This shows the claim. □

Using the Lemma 1.16 above one easily sees that there exists a (non-unique) $\beta_A \in \delta_A + \mathbb{Q}_+ A \cap \mathbb{Z}^d$ such that $\mathfrak{F}_A := \beta_A + ([0,1)^d \cap \mathbb{Q}^d)$ is contained in $\delta_A + \mathbb{Q}_+ A$. The following lemma shows that this set $\mathfrak{F}_A$ can be considered as a fundamental domain for rational β which are not strongly resonant.

Lemma 1.18. *Let $\beta \in \mathbb{Q}^d$ with $\beta \notin sRes(A)$, then there exist a $\beta' \in \mathfrak{F}_A$ such that $\mathcal{M}_A^\beta \simeq \mathcal{M}_A^{\beta'}$.*

Proof. Notice that there exist a $\alpha \in \mathbb{Z}^d$ so that $\beta + \alpha \in \mathfrak{F}_A$. We have

$$\mathrm{FL}\, \mathcal{M}_A^\beta \simeq (k \circ l)_+ \mathcal{O}_{T'}^\beta \simeq (k \circ l)_+ \mathcal{O}_{T'}^{\beta+\alpha} \simeq \mathrm{FL}(\mathcal{M}_A^{\beta+\alpha}) \, ,$$

where we have used that $\mathcal{O}_{T'} \underline{y}^\beta \simeq \mathcal{O}_{T'} \underline{y}^{\beta+\alpha}$. Setting $\beta' := \beta + \gamma$ shows the claim. □

In the rest of this section we will consider special types of GKZ-system. Let A be a $d \times n$-matrix with $\mathbb{Z}A = \mathbb{Z}^d$. We define a $d+1 \times n+1$-matrix

$$\widetilde{A} := \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix} .$$

We will consider the GKZ-system $\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}}$ with $\widetilde{\beta} = (\beta_0, \dots, \beta_d) \in \mathbb{C}^{d+1}$ and denote the coordinates of the underlying space as $\lambda_0, \dots, \lambda_m$.
Notice that the semigroup $\mathbb{N}\widetilde{A}$ is always pointed, thus every statement above applies to these kind of matrices.

Later we will need the following lemma.

Lemma 1.19. *Let A be a $d \times n$ integer matrix with $\mathbb{Z}A = \mathbb{Z}^d$. If $\beta \notin sRes(A)$, there exist a $n_\beta \in \mathbb{Z}$ such that $\widetilde{\beta} = (\beta_0, \beta) \notin sRes(\widetilde{A})$ for all $\beta_0 \in \mathbb{Q}$ with $\beta_0 \geq n_\beta$.*

Proof. Fix a $\beta \in \mathbb{Q}^d$. At first we prove that there exists an $n_\beta \in \mathbb{Z}$, s.t. for $\beta_0 \geq n_\beta$ the element $(\beta_0, \beta) \notin sRes_0(\widetilde{A})$. For this we have to compute the quasi-degrees $\mathbb{Q}^{d+1} \cap qdeg(S_{\widetilde{A}}/\partial_{\lambda_0})$. Recall that $\mathbb{Q}^{d+1} \cap qdeg(S_{\widetilde{A}}/\partial_{\lambda_0})$ is a finite union of translates of $\mathbb{Q}$-linear spans of faces of $\mathbb{Q}_+\widetilde{A}$ which do not contain $\underline{\widetilde{a}}_0 = (1, 0, \dots, 0)$ (cf.(1.4.3)). Thus we can find an $n_\beta \in \mathbb{Z}$ so that for every $\beta_0 \geq n_\beta$ the element $(\beta_0, \beta) \notin sRes_0(\widetilde{A})$.
Now assume additionally that $\beta \notin sRes_j(A)$ for some $j \in \{1, \dots, n\}$. Recall that this means

$$\beta \notin -(\mathbb{N}+1)\underline{a}_j - qdeg(S_A/\partial_{\lambda_j}).$$

Notice that $qdeg(S_{\widetilde{A}}/\partial_{\lambda_j}) \subset \mathbb{Q} \times qdeg(S_A/\partial_{\lambda_j})$. But this means

$$\begin{aligned} sRes_j(\widetilde{A}) &= -(\mathbb{N}+1)\underline{\widetilde{a}}_j - qdeg(S_{\widetilde{A}}/\partial_{\lambda_j}) \\ &\subset -(\mathbb{N}+1)(\mathbb{Q} \times \underline{a}_j) - (\mathbb{Q} \times qdeg(S_A/\partial_{\lambda_j})) \\ &= \mathbb{Q} \times sRes_j(A) \, . \end{aligned}$$

But this means that $(\beta_0, \beta) \notin sRes_j(\widetilde{A})$ for any $\beta_0 \in \mathbb{Q}$. Summarizing we have shown that if $\beta \notin sRes(A)$, then $(\beta_0, \beta) \notin sRes(\widetilde{A})$ for any $\beta_0 \in \mathbb{Q}$ with $\beta_0 \geq n_\beta$, but this shows the claim. $\square$

1.4.2 Isomorphism classes of GKZ-systems

As a next step we want to describe isomorphism classes of GKZ systems $\mathcal{M}^{\widetilde{\beta}}_{\widetilde{A}}$, i.e. given a $\widetilde{\beta}$, we want to determine for which $\widetilde{\beta}'$ the GKZ-systems $\mathcal{M}^{\widetilde{\beta}}$ and $\mathcal{M}^{\widetilde{\beta}'}_{\widetilde{A}}$ are isomorphic. The first important observation is, that a necessary condition for this isomorphism to hold is $\widetilde{\beta} - \widetilde{\beta}' \in \mathbb{Z}^{d+1}$ [Sai01, proposition 2.2 1.]. In the following we denote by $(\mathbb{Q}_+\widetilde{A})^\circ$ the set $\mathbb{Q}_+\widetilde{A} \cap (\mathbb{R}_+\widetilde{A})^\circ$ where $(\mathbb{R}_+\widetilde{A})^\circ$ is the topological interior of $\mathbb{R}_+\widetilde{A} \subset \mathbb{R}^{d+1}$.

Lemma 1.20.

1. *Let $\widetilde{\beta}, \widetilde{\beta}' \in -(\mathbb{Q}_+\widetilde{A})^\circ$, then $\mathcal{M}^{\widetilde{\beta}}_{\widetilde{A}} \simeq \mathcal{M}^{\widetilde{\beta}'}_{\widetilde{A}}$ if and only if $\widetilde{\beta} - \widetilde{\beta}' \in \mathbb{Z}\widetilde{A}$.*
2. *Let $\widetilde{\beta}, \widetilde{\beta}' \in \delta_{\widetilde{A}} + \mathbb{Q}_+\widetilde{A}$, then $\mathcal{M}^{\widetilde{\beta}}_{\widetilde{A}} \simeq \mathcal{M}^{\widetilde{\beta}'}_{\widetilde{A}}$ if and only if $\widetilde{\beta} - \widetilde{\beta}' \in \mathbb{Z}\widetilde{A}$.*
3. *If $\mathbb{N}\widetilde{A}$ is saturated and $\beta, \beta' \in \mathbb{Q}_+\widetilde{A}$, then $\mathcal{M}^{\beta}_{\widetilde{A}} \simeq \mathcal{M}^{\beta'}_{\widetilde{A}}$ if and only if $\beta - \beta' \in \mathbb{Z}\widetilde{A}$.*

Proof. The first point follows from [Sai01] Corollary 2.6. The second resp. third point follows from Theorem 1.15 and Lemma 1.16 resp. Lemma 1.17. $\square$

Later we will be especially interested in the case $\widetilde{\beta} \in \mathbb{Z}^{d+1}$. We need the following definitions of [Sai01]. Let σ be a facet of $\mathbb{Q}_+\widetilde{A}$, i.e. a codimenion one face. To each facet we associate a unique primitive, integral support function F_σ satisfying the following properties

1. $F_\sigma(\mathbb{Z}\widetilde{A}) = \mathbb{Z}$,
2. $F_\sigma(\widetilde{a}_j) \geq 0$ for all $j = 0, \ldots, n$,
3. $F_\sigma(\widetilde{a}_j) = 0$ for all $\widetilde{a}_j \in \sigma$.

We state a special case of a classification theorem of [Sai01].

Theorem 1.21. *[Sai01, Theorem 5.2] Let $\mathbb{N}\widetilde{A}$ be saturated and $\widetilde{\beta}, \widetilde{\beta}' \in \mathbb{Z}^{d+1}$. Then $\mathcal{M}^{\widetilde{\beta}}_{\widetilde{A}} \simeq \mathcal{M}^{\widetilde{\beta}'}_{\widetilde{A}}$ if and only if*

$$\{\sigma \mid F_\sigma(\widetilde{\beta}) \in \mathbb{N}\} = \{\sigma \mid F_\sigma(\widetilde{\beta}') \in \mathbb{N}\}.$$

1.4.3 Holonomic duals of GKZ-systems

In this section we want to present results of Walther on the holonomic duals of GKZ-systems. Takayama's conjecture states that the holonomic dual of a GKZ-system is again a GKZ-system. Walther showed that this is in general wrong, but it is true for generic parameters β.

Theorem 1.22. *[Wal07, Theorem 4.8] For all $\widetilde{A}$ there is a Zariski open subset $U \subset \mathbb{C}^{d+1}$ of parameters such that $\mathbb{D}\mathcal{M}^{\beta}_{\widetilde{A}}$ is isomorphic to $\mathcal{M}^{-\beta-\epsilon_{\widetilde{A}'}}_{\widetilde{A}}$ for all $\beta \in U$.*

Here $\epsilon_{\widetilde{A}'}$ is the sum of the columns, seen as elements of $\mathbb{Z}^{d+1}$ of a matrix $\widetilde{A}'$, where $\widetilde{A}'$ is a matrix which generates the saturation of $\mathbb{N}\widetilde{A}$. In particular $\epsilon_{\widetilde{A}'}$ is in $\mathbb{Q}_+\widetilde{A} \cap \mathbb{Z}\widetilde{A}$. We deduce the following statement from the theorem above.

Proposition 1.23. *Let $\delta_{\widetilde{A}} \in \mathbb{N}\widetilde{A}$ as in Lemma 1.16 and $\beta \in \delta_{\widetilde{A}} + \mathbb{R}_+\widetilde{A}$. Then*

$$\mathbb{D}\mathcal{M}^{\beta}_{\widetilde{A}} \simeq \mathcal{M}^{-\beta+\kappa}_{\widetilde{A}}$$

for all $\kappa \in \mathbb{Z}^{d+1}$ for which $-\beta + \kappa \in -(\mathbb{R}_+\widetilde{A})^\circ$ holds. If $\mathbb{N}\widetilde{A}$ is saturated, the theorem holds with $\delta_{\widetilde{A}} = 0$.

Proof. Let $\beta \in \delta_{\widetilde{A}} + \mathbb{R}_+\widetilde{A}$. There exists an $\alpha \in \mathbb{Z}^{d+1}$ such that $\beta + \alpha \in U \cap \delta_{\widetilde{A}} + \mathbb{R}_+\widetilde{A}$. We have the following isomorphisms

$$\mathbb{D}\mathcal{M}^{\beta}_{\widetilde{A}} \simeq \mathbb{D}\mathcal{M}^{\beta+\alpha}_{\widetilde{A}} \simeq \mathcal{M}^{-\beta-\alpha-\epsilon_{\widetilde{A}'}}_{\widetilde{A}} \simeq \mathcal{M}^{-\beta+\gamma}_{\widetilde{A}}$$

for all $\gamma \in \mathbb{Z}^{d+1}$ with $-\beta + \gamma \in -(\mathbb{R}_+\widetilde{A})^\circ$. The first isomorphism holds because of Lemma 1.20 2. and the last isomorphism holds because $-\beta - \alpha - \epsilon_{\widetilde{A}'} \in -(\mathbb{R}_+ A)^\circ$ and Lemma 1.20 1. . The last statement follows from Lemma 1.20 3. . ☐

1.4.4 Morphisms of Hypergeometric Systems

We need still another fact of the theory of hypergeometric systems $\mathcal{M}^{\widetilde{\beta}}_{\widetilde{A}}$.
Let $\widetilde{\beta}, \widetilde{\beta}' \in \mathbb{C}^{d+1}$ so that $\widetilde{\beta} - \widetilde{\beta}' \in \mathbb{N}\widetilde{A}$. There is the following rigidity result for morphisms between such GKZ-systems.

Proposition 1.24. *Let $\psi : \mathcal{M}^{\widetilde{\beta}'}_{\widetilde{A}} \longrightarrow \mathcal{M}^{\widetilde{\beta}}_{\widetilde{A}}$ be a non-zero $\mathcal{D}$-linear morphism, then (up to multiplication with a non-zero constant) ψ is given by right multiplication with $\partial^{\beta-\beta'}$.*

Proof. In the course of the proof we will work with the modules of global sections instead of the $\mathcal{D}$-modules themselves. First notice that ψ is determined by the image of $[1] \in M^{\beta'}_{\widetilde{A}}$ because $M^{\beta'}_{\widetilde{A}}$ is a cyclic left D-module. Let $P \in D$ so that $\psi([1]) = [P]$. Define for $\widetilde{\kappa} \in \mathbb{C}^{d+1}$ the $\mathbb{C}[\lambda_0\partial_{\lambda_0}, \ldots, \lambda_n\partial_{\lambda_n}]$-module

$$(M^{\widetilde{\beta}}_{\widetilde{A}})_{\widetilde{\kappa}} := \{[Q] \in M^{\widetilde{\beta}}_{\widetilde{A}} \mid (E_k + \kappa_k) \cdot [Q] = 0 \text{ for all } k \in \{0, \ldots, d\}\} .$$

From [Sai07, Chaper 4] follows that there exists the following $\mathbb{C}[\lambda_0\partial_{\lambda_0}, \ldots, \lambda_n\partial_{\lambda_n}]$-module isomorphism for $\widetilde{\beta} + \widetilde{\kappa} \in \mathbb{N}\widetilde{A}$:

$$\begin{aligned} \mathbb{C}[\lambda_0\partial_{\lambda_0}, \ldots, \lambda_n\partial_{\lambda_n}]/\sum_{k=0}^{d}(E_k + \kappa_k)\mathbb{C}[\lambda_0\partial_{\lambda_0}, \ldots, \lambda_n\partial_{\lambda_n}] &\longrightarrow (M^{\widetilde{\beta}}_{\widetilde{A}})_{\widetilde{\kappa}} \\ g(\lambda\partial_\lambda) := g(\lambda_0\partial_{\lambda_0}, \ldots, \lambda_n\partial_{\lambda_n}) &\mapsto g(\lambda\partial_\lambda)\partial^{\widetilde{\beta}+\widetilde{\kappa}} \end{aligned} \tag{1.4.4}$$

In order that ψ is well-defined, we must have $[P] \in (M^{\widetilde{\beta}}_{\widetilde{A}})_{-\widetilde{\beta}'}$. Therefore P can be chosen to be in $\mathbb{C}[\lambda_0\partial_{\lambda_0}, \ldots, \lambda_n\partial_{\lambda_n}]\partial^{\widetilde{\beta}-\widetilde{\beta}'}$. We will write $\psi([1]) = [f(\lambda\partial_\lambda)\partial^{\widetilde{\beta}-\widetilde{\beta}'}]$ for short and compute the image of $\Box_{\underline{l}} = \partial^{l^-} - \partial^{l^+}$

$$\begin{aligned} 0 &= \psi([\partial^{l^-} - \partial^{l^+}]) \\ &= (\partial^{l^-} - \partial^{l^+}) \cdot \psi([1]) = (\partial^{l^-} - \partial^{l^+})[f(\lambda\partial_\lambda)\partial^{\widetilde{\beta}-\widetilde{\beta}'}] \\ &= [(f(\lambda\partial_\lambda + l^-)\partial^{l^-} - f(\lambda\partial_\lambda + l^+)\partial^{l^+})\partial^{\widetilde{\beta}-\widetilde{\beta}'}] \\ &= [(f(\lambda\partial_\lambda + l^-) - f(\lambda\partial_\lambda + l^+))\partial^{l^+}\partial^{\widetilde{\beta}-\widetilde{\beta}'}] \\ &\in (M^{\widetilde{\beta}}_{\widetilde{A}})_{\widetilde{A}l^+-\widetilde{\beta}'} , \end{aligned} \tag{1.4.5}$$

where $\widetilde{A}l^+ \in \mathbb{Z}^{d+1}$ with k-th component $(\widetilde{A}l^+)_k := \sum_{i=0}^n \widetilde{a}_{ki}(l^+)_i$. Because of (1.4.4) we have

$$f(\lambda\partial_\lambda + l^-) - f(\lambda\partial_\lambda + l^+) \in \sum_{k=0}^{d}(E_k + (\widetilde{A}l^+)_k - \widetilde{\beta}'_k)\mathbb{C}[\lambda_0\partial_{\lambda_0}, \ldots, \lambda_n\partial_{\lambda_n}] .$$

Notice that for a relation $l \in \mathbb{N}^{n+1}$ we have

$$f(\lambda\partial_\lambda) - f(\lambda\partial_\lambda + l) \in \sum_{k=0}^{d}(E_k - \widetilde{\beta}'_k)\mathbb{C}[\lambda_0\partial_{\lambda_0}, \ldots, \lambda_n\partial_{\lambda_n}] .$$

The statement above is a statement in a commutative ring. For better readability we set $x_i = \lambda_i\partial_{\lambda_i}$, then the statement above can be expressed as (recall that $\widetilde{E}_k = \sum_{i=0}^n \widetilde{a}_{ki}\lambda_i\partial_{\lambda_i}$)

$$f(x) - f(x+l) \in \sum_{k=0}^{d}((\widetilde{A}\cdot x)_k - \widetilde{\beta}'_k)\mathbb{C}[x]\,.$$

Because the columns of $\widetilde{A}$ generate $\mathbb{Z}^{d+1}$, i.e $\widetilde{A}$ has full rank, we can find a $\gamma \in \mathbb{C}^{d+1}$ with $\widetilde{A}\cdot\gamma = \widetilde{\beta}'$. Thus we have

$$f(\gamma) - f(\gamma + l) = 0\,.$$

Since f is a polynomial this means that f has constant value $f(\gamma)$ one the affine subspace $\gamma + \ker(\widetilde{A})$. But this means

$$f(\gamma + x) \in f(\gamma) + \sum_{k=0}^{d}(\widetilde{A}\cdot x)_k\mathbb{C}[x]\,,$$

resp.

$$f(x) \in f(\gamma) + \sum_{k=0}^{d}((\widetilde{A}\cdot x)_k - \widetilde{\beta}'_k)\mathbb{C}[x]\,.$$

If we substitute $\lambda\partial_\lambda$ back, we get

$$\psi([1]) = [P] = [f(\lambda\partial_\lambda)\partial^{\widetilde{\beta}-\widetilde{\beta}'}] = [f(\gamma)\partial^{\widetilde{\beta}-\widetilde{\beta}'}]\,,$$

where we have used $(E_k - \widetilde{\beta}'_k)\partial^{\widetilde{\beta}-\widetilde{\beta}'} = \partial^{\widetilde{\beta}-\widetilde{\beta}'}(E_k - \widetilde{\beta}_k)$. This shows the claim. □

1.4.5 Characteristic varieties of GKZ systems

Let $\widetilde{A}$ be the $d+1\times n+1$ matrix as above. Let $\widetilde{Q}$ be the convex hull of $\widetilde{a}_0,\ldots,\widetilde{a}_n$ in $\mathbb{R}^{d+1}$. Notice that $\widetilde{Q} \subset \{1\}\times\mathbb{R}^d$ and therefore no face Γ of $\widetilde{Q}$ contains the origin. Adolphson characterized the characteristic variety $char(\mathcal{M}^{\widetilde{\beta}}_{\widetilde{A}})$ of the GKZ system $\mathcal{M}^{\widetilde{\beta}}_{\widetilde{A}}$ as follows. Let $T^*V \simeq V\times V'$ be the holomorphic cotangent bundle with coordinates $(\lambda_0,\ldots,\lambda_n,\mu_0,\ldots,\mu_n)$. Define the following Laurent polynomials on $(\mathbb{C}^*)^{d+1}$

$$f_{\underline{\lambda}}(\underline{y}) := \sum_{i=0}^{n}\lambda_i\underline{y}^{\underline{\widetilde{a}}_i}\,,$$
$$f_{\underline{\lambda},\Gamma}(\underline{y}) := \sum_{\underline{\widetilde{a}}_i\in\Gamma}\lambda_i\underline{y}^{\underline{\widetilde{a}}_i}\,,$$

where we define $\underline{y}^{\underline{\widetilde{a}}_i} := \prod_{k=0}^n y_k^{\widetilde{a}_{ki}}$.

Lemma 1.25 ([Ado94] Lemma 3.2, Lemma 3.3)**.**

1. *For each $(\underline{\lambda}^{(0)},\underline{\mu}^{(0)}) \in char(\mathcal{M}^{\widetilde{\beta}}_{\widetilde{A}})$ there exists a (possibly empty) face Γ such that $\mu_j^{(0)} \neq 0$ if and only if $\underline{\widetilde{a}}_j \in \Gamma$.*
2. *If $\underline{\lambda}^{(0)}$ is a singular point of $\mathcal{M}^{\widetilde{\beta}}_{\widetilde{A}}$ and Γ the corresponding (non-empty) face, then the Laurent polynomials $\partial f_{\underline{\lambda}^{(0)},\Gamma}/\partial y_0,\ldots,\partial f_{\underline{\lambda}^{(0)},\Gamma}/\partial y_n$ have a common zero in $(\mathbb{C}^*)^{d+1}$.*

2 Families of Laurent Polynomials and Hypergeometric $\mathcal{D}$-modules

In this section we will prove the relationship between the Gauß-Manin system of a family of Laurent polynomials and a certain direct sum of hypergeometric systems.
Let B be a $d\times n$-matrix, where we denote the columns by $\underline{b}_1,\ldots,\underline{b}_n$. We assume that the columns of B generate $\mathbb{Q}^d$. We associate to this matrix the following family of Laurent polynomials

$$\begin{aligned}\varphi_B : S\times W &\longrightarrow \mathbb{C}_{\lambda_0}\times W\,,\\ (y_1,\ldots,y_d,\lambda_1,\ldots,\lambda_n) &\mapsto (-\sum_{i=1}^{n}\lambda_i \underline{y}^{\underline{b}_i},\lambda_1,\ldots,\lambda_n)\,,\end{aligned}$$

where $S:=(\mathbb{C}^*)^d$, $W:=\mathbb{C}^n$.
We will construct a morphism with $\mathcal{O}$-free kernel and cokernel between the Gauß-Manin System $\mathcal{H}^0(\varphi_{B+}\mathcal{O}_{S\times W})$ and a direct sum of GKZ-systems. We will first reduce to the case where the columns of the matrix B generate $\mathbb{Z}^d$.

Using the Smith normal form we can write B as

$$C\cdot D_1\cdot D_2\cdot M$$

with $C=(\underline{c}_1,\ldots,\underline{c}_d)\in Gl(d\times d,\mathbb{Z}), M=(\underline{m}_1,\ldots,\underline{m}_n)\in GL(n\times n,\mathbb{Z})$ and

$$D_1\cdot D_2 = \begin{pmatrix} e_1 & & \\ & \ddots & \\ & & e_d\end{pmatrix}\cdot\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0\\ & & 1 & \end{array}\right).$$

Hence the family of Laurent polynomials φ_B can be factored into an isomorphism

$$\begin{aligned}\vartheta : S\times W &\longrightarrow S\times W\,,\\ (y_1,\ldots,y_d,\lambda_1,\ldots,\lambda_n) &\mapsto (\underline{y}^{\underline{c}_1},\ldots,\underline{y}^{\underline{c}_d},\lambda_1,\ldots,\lambda_n)\end{aligned}$$

followed by the family of Laurent polynomials associated to the matrix $B'=(\underline{b}'_1,\ldots,\underline{b}'_n):=D_1\cdot D_2\cdot M$:

$$\begin{aligned}\varphi_{B'} : S\times W &\longrightarrow \mathbb{C}_{\lambda_0}\times W\,,\\ (y_1,\ldots,y_d,\lambda_1,\ldots,\lambda_n) &\mapsto (-\sum_{i=1}^{n}\lambda_i \underline{y}^{\underline{b}'_i},\lambda_1,\ldots,\lambda_n)\,.\end{aligned}$$

We state the following lemma for further reference.

Lemma 2.1. *Let the matrices B, B' as above. Then*

$$\varphi_{B,+}\mathcal{O}_{S\times W}\simeq \varphi_{B',+}\mathcal{O}_{S\times W}\,.$$

Proof. This follows from $\vartheta_+\mathcal{O}_{S\times W}\simeq\mathcal{O}_{S\times W}$. ☐

Let $A:=D_2\cdot M$ and set $\widetilde{A}:=(\underline{\widetilde{a}}_0,\underline{\widetilde{a}}_1,\ldots,\underline{\widetilde{a}}_n)$ with $\underline{\widetilde{a}}_0:=(1,\underline{0})$ and $\underline{\widetilde{a}}_i:=(1,\underline{a}_i)$ for $i=1,\ldots,n$.

Consider the following maps

$$\begin{aligned}\pi^e : S &\longrightarrow S\\ (y_1,\ldots,y_d) &\mapsto (y_1^{e_1},\ldots,y_d^{e_d})\end{aligned}$$

and

$$
\begin{aligned}
g : S &\longrightarrow \mathbb{P}^n \,, \\
(y_1, \ldots, y_d) &\mapsto (1 : \underline{y}^{\underline{a}_1} : \ldots : \underline{y}^{\underline{a}_n}) \,,
\end{aligned}
\tag{2.0.6}
$$

and define $g^e := g \circ \pi^e$.

In order to construct the morphisms between the (proper) Gauß-Manin system of φ_B resp. $\varphi_{B'}$ and the direct sum of GKZ-systems, consider the following exact triangles in $D^b_{rh}(V)$ from Proposition 1.9:

$$
\begin{aligned}
\mathcal{R}(g^e_\dagger\, \mathcal{O}_S)[-1] \longrightarrow \mathcal{R}_{cst}(g^e_\dagger\, \mathcal{O}_S) \longrightarrow \mathcal{R}^\circ(g^e_\dagger\, \mathcal{O}_S) \xrightarrow{+1} \\
\mathcal{R}^\circ_c(g^e_+\, \mathcal{O}_S) \longrightarrow \mathcal{R}_{cst}(g^e_+\, \mathcal{O}_S) \longrightarrow \mathcal{R}(g^e_+\, \mathcal{O}_S)[1] \xrightarrow{+1} \,,
\end{aligned}
\tag{2.0.7}
$$

where $\mathcal{O}_S := a^+_S \mathcal{O}_{pt}$. In the following pages we will calculate each term of the triangles above.

Consider the following maps

$$
\begin{aligned}
\widetilde{\pi}^e : T &\longrightarrow T \\
(y_0, y_1, \ldots, y_d) &\mapsto (y_0, y_1^{e_1}, \ldots, y_d^{e_d})
\end{aligned}
$$

and

$$
\begin{aligned}
h : T &\longrightarrow V' \\
(y_0, \ldots, y_d) &\mapsto (\underline{y}^{\underline{\widetilde{a}}_0}, \ldots, \underline{y}^{\underline{\widetilde{a}}_n}) \,,
\end{aligned}
$$

where $T \simeq (\mathbb{C}^*)^{d+1}$, $V' \simeq \mathbb{C}^{n+1}$ and h corresponds to the map (1.4.1) associated to the matrix $\widetilde{A}$. Define $h^e := h \circ \widetilde{\pi}^e$.

These maps give rise to the following diagram, where the lower square is cartesian and the upper left triangle is commutative.

$$
\begin{tikzcd}
 & V' & Bl_0(V') \arrow[l, "p"'] \arrow[ddl, "q"] \\
T \arrow[ur, "h^e"] \arrow[r, "\widetilde{h}^e"'] \arrow[d, "\pi_T"'] & V' \setminus \{0\} \arrow[u, "j"'] \arrow[d, "\pi"'] & \\
S \arrow[r, "g^e"] & \mathbb{P}(V') &
\end{tikzcd}
.
$$

Here, the maps π and π_T are the canonical projections and j is the canonical inclusion. Finally $p : Bl_0(V') \longrightarrow V'$ is the blowup of $0 \in V'$ and $q : Bl_0(V') \subset \mathbb{P}(V') \times V' \longrightarrow \mathbb{P}(V')$ is the projection on the first factor.

In the next lemma we compare various $\mathcal{D}$-modules constructed from the $\mathcal{D}$-modules $\mathcal{O}_S$ resp. $\mathcal{O}_T := a^+_T \mathcal{O}_{pt}$, living on the d-dimensional torus S resp. $d+1$-dimensional torus T.

Lemma 2.2.

1. *The functors h^e_+ and $h^e_\dagger$ are exact.*
2. *We have isomorphisms*

$$
\begin{aligned}
j_+ \pi^+ g^e_+ \mathcal{O}_S &\simeq h^e_+ \mathcal{O}_T \,, \\
j_\dagger \pi^+ g^e_\dagger \mathcal{O}_S &\simeq h^e_\dagger \mathcal{O}_T \,,
\end{aligned}
$$

in the category of monodromic, $\mathcal{D}$-modules on V'.

3. *The (shifted) $\mathcal{D}$-modules $h_+\mathcal{O}_T$ and $h_\dagger\mathcal{O}_T$ are monodromic on $V' \cong \mathbb{C}^{n+1}$.*

Proof.

1. Recall that $h^e = h \circ \widetilde{\pi}^e$. The exactness of h_+ follows from the fact that the map h is an affine embedding and from [BGK+87, VI,Proposition 8.1]. The exactness of $\widetilde{\pi}^e_+$ follows again from [BGK+87, VI, Proposition 8.1] and the fact that $\widetilde{\pi}^e$ is quasi-finite. This shows the exactness of h^e_+, the exactness of $h^e_\dagger$ follows by duality.
2. To prove the second point, observe that the following diagram is cartesian:

$$\begin{tikzcd} T \arrow[r, "\widetilde{h}^e"] \arrow[d, "\pi_T"'] & V' \setminus \{0\} \arrow[d, "\pi"] \\ S \arrow[r, "g^e"] & \mathbb{P}(V')\,. \end{tikzcd}$$

   Using base change with respect to π (cf. Remark 1.4) we get

$$\pi^+ g^e_+ \mathcal{O}_S \simeq \widetilde{h}^e_+ \pi^+_T \mathcal{O}_S \simeq \widetilde{h}^e_+ \mathcal{O}_T\,,$$

   where we have used that $\pi^+_T\, \mathcal{O}_S \simeq \mathcal{O}_T$.

   From this follows

$$h_+\mathcal{O}_T \simeq j_+\widetilde{h}_+\mathcal{O}_T \simeq j_+\pi^+ g_+\mathcal{O}_S\,.$$

   The second isomorphism follows by duality and the fact that $\pi^\dagger = \pi^+$.
3. The fact that $h_+\mathcal{O}_T$ and $h_\dagger\mathcal{O}_T$ are monodromic follows from the isomorphisms in (2) and Remark 1.8.

□

In the next proposition we compare the direct image of $\mathcal{O}_{S\times W}$ under φ_A with the Radon transform of $g_+\mathcal{O}^\beta_S$. Here and in the following we will identify V with $\mathbb{C}_{\lambda_0} \times W$.

Proposition 2.3.

1. *Let $\varphi_A : S \times W \to \mathbb{C}_{\lambda_0} \times W$ be the family of Laurent polynomials defined above. Then we have the following isomorphisms in $D^b_{rh}(\mathcal{D}_V)$*

$$\mathcal{R}(g^e_+\mathcal{O}_S) \simeq \varphi_{B',+}\mathcal{O}_{S\times W}\,,$$
$$\mathcal{R}(g^e_\dagger\mathcal{O}_S) \simeq \varphi_{B',\dagger}\mathcal{O}_{S\times W}\,.$$

2. *There are isomorphisms*

$$\mathcal{H}^i(\mathcal{R}_{cst}(g^e_+\mathcal{O}_S)) \simeq H^{d+i}(S,\mathbb{C}) \otimes \mathcal{O}_V\,,$$
$$\mathcal{H}^i(\mathcal{R}_{cst}(g^e_\dagger\mathcal{O}_S)) \simeq H_{d-i}(S,\mathbb{C}) \otimes \mathcal{O}_V\,.$$

   in $D^b_{rh}(\mathcal{D}_V)$.
3. *There are isomorphisms*

$$\mathcal{R}^\circ_c(g^e_+\mathcal{O}_S) \simeq \mathrm{FL}(h^e_+\mathcal{O}_T)\,,$$
$$\mathcal{R}^\circ(g^e_\dagger\mathcal{O}_S) \simeq \mathrm{FL}(h^e_\dagger\mathcal{O}_T)$$

   in $D^b_{rh}(\mathcal{D}_V)$.

Proof. 1. Consider the following diagram, where the square is cartesian

$$\begin{tikzcd} S\times W \arrow[r, "\widetilde{i}"] & \Gamma \arrow[r, "\kappa"] \arrow[d, "\eta"'] & Z \arrow[r, "\pi_2^Z"'] \arrow[d, "\pi_1^Z"] & V \\ & S \arrow[r, "g^e"] & \mathbb{P}(V') & \end{tikzcd} .$$

The hypersurface Z in $\mathbb{P}(V')\times V$ is given by $\sum_{i=0}^n \lambda_i\mu_i = 0$ and Γ is the fibered product of S and Z. The map $g^e : S \to \mathbb{P}(V')$ is given by

$$\begin{aligned} g^e = g\circ\pi^e : S &\longrightarrow \mathbb{P}(V')\,, \\ (y_1,\ldots,y_d) &\mapsto (1 : \underline{y}^{b'_1} : \ldots : \underline{y}^{b'_n})\,. \end{aligned}$$

Thus Γ is the smooth hypersurface in $S\times V$ given by $\lambda_0 + \sum_{i=1}^n \lambda_i \underline{y}^{b'_i} = 0$. Notice that we have an isomorphism $\widetilde{i} : S\times W \longrightarrow \Gamma$ given by

$$(y_1,\ldots,y_d,\lambda_1,\ldots,\lambda_n) \mapsto (y_1,\ldots,y_d, -\sum_{i=1}^n \lambda_i\underline{y}^{b'_i}, \lambda_1,\ldots,\lambda_n)\,.$$

We have

$$\mathcal{R}(g^e_+\mathcal{O}_S) \simeq \pi^Z_{2+}\,(\pi^Z_1)^+\, g^e_+\,\mathcal{O}_S \simeq \pi^Z_{2+}\kappa_+\,\eta^+\,\mathcal{O}_S$$

by base change with respect to the map π_1^Z and Remark 1.4.
Notice that $\pi_2^Z\circ\kappa\circ\widetilde{i} = \varphi_{B'}$ by the definition of Γ. Using the fact that $\eta^+\,\mathcal{O}_S \simeq \widetilde{i}_+\mathcal{O}_{S\times W}$, we obtain

$$\mathcal{R}(g^e_+\mathcal{O}_S) \simeq \varphi_{B',+}\,\mathcal{O}_{S\times W}\,.$$

The second statement is just the dual statement of the first.

2. Consider the cartesian diagram

$$\begin{tikzcd} & \mathbb{P}(V')\times V \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}(V') \arrow[dr, "a_{\mathbb{P}}"'] & & V \arrow[dl, "a_V"] \\ & pt & \end{tikzcd} .$$

We have

$$\mathcal{R}_{cst}(g^e_+\mathcal{O}_S) \simeq \pi_{2+}\,\pi_1^+\,g^e_+\mathcal{O}_S \simeq a_V^+\,a_{\mathbb{P}+}\,g^e_+\mathcal{O}_S \simeq a_V^+ a_{S+}\mathcal{O}_S$$

by base change with respect to a_V and Remark 1.4. We get

$$\mathcal{H}^i\mathcal{R}_{cst}(g^e_+\mathcal{O}_S) \simeq H^{d+i}(S,\mathbb{C})\otimes\mathcal{O}_V\,.$$

For the second statement we have $\mathcal{R}_{cst}(g^e_\dagger) \simeq a_V^+ a_{S\dagger}\mathcal{O}_S$, which gives

$$\mathcal{H}^i\mathcal{R}_{cst}(g^e_\dagger\mathcal{O}_S) \simeq H^{d+i}_c(S,\mathbb{C})\otimes\mathcal{O}_V \simeq H_{d-i}(S,\mathbb{C})\otimes\mathcal{O}_V\,,$$

where the last isomorphism follows from Poincaré-Verdier duality.

3. The first statement follows from the second isomorphism in Proposition 1.10 and 2.2 (2). The second statement is again the dual of the first.

□

Recall the following triangles in $D^b_{rh}(V)$ from above:

$$\begin{aligned} \mathcal{R}(g^e_\dagger\,\mathcal{O}_S)[-1] &\longrightarrow \mathcal{R}_{cst}(g^e_\dagger\,\mathcal{O}_S) \longrightarrow \mathcal{R}^\circ(g^e_\dagger\,\mathcal{O}_S) \xrightarrow{+1}, \\ \mathcal{R}^\circ_c(g^e_+\,\mathcal{O}_S) &\longrightarrow \mathcal{R}_{cst}(g^e_+\,\mathcal{O}_S) \longrightarrow \mathcal{R}(g^e_+\,\mathcal{O}_S)[1] \xrightarrow{+1}. \end{aligned} \tag{2.0.8}$$

Using Proposition 2.3, this will enable us to extract information about the cohomology of the (proper) direct image of $\varphi_{B'}$.

Proposition 2.4. *Let $\varphi_{B'} : S \times W \longrightarrow V$ be the family of Laurent polynomials defined above.*

1. *$\mathcal{H}^k(\varphi_{B',+}\mathcal{O}_{S\times W}) = 0$ for $k \notin [-d+1, 0]$ and $\mathcal{H}^k(\varphi_{B',\dagger}\mathcal{O}_{S\times W}) = 0$ for $k \notin [0, d-1]$.*
2. *$\mathcal{H}^k(\varphi_{B',+}\mathcal{O}_{S\times W})$ is isomorphic to the free $\mathcal{O}_V$-module $H^{d-1+k}(S,\mathbb{C}) \otimes \mathcal{O}_V$ for $k \in [-d+1, -1]$ and*
   $\mathcal{H}^k(\varphi_{A,\dagger}\mathcal{O}_{S\times W})$ is isomorphic to the free $\mathcal{O}_V$-module $H_{d-1-k}(S,\mathbb{C}) \otimes \mathcal{O}_V$ for $k \in [1, d-1]$.
3. *There are the following exact sequences in $M_{rh}(\mathcal{D}_V)$:*

$$0 \longrightarrow H^{d-1}(S,\mathbb{C}) \otimes \mathcal{O}_V \longrightarrow \mathcal{H}^0(\varphi_{B',+}\mathcal{O}_{S\times W}) \longrightarrow \mathrm{FL}(h^e_+\mathcal{O}_T) \longrightarrow H^d(S,\mathbb{C}) \otimes \mathcal{O}_V \longrightarrow 0\,,$$
$$0 \longrightarrow H_d(S,\mathbb{C}) \otimes \mathcal{O}_V \longrightarrow \mathrm{FL}(h^e_\dagger\mathcal{O}_T) \longrightarrow \mathcal{H}^0(\varphi_{B',\dagger}\mathcal{O}_{S\times W}) \longrightarrow H_{d-1}(S,\mathbb{C}) \otimes \mathcal{O}_V \longrightarrow 0\,.$$

Proof. Notice that we have

$$\mathcal{H}^i\mathcal{R}^\circ_c(g^e_+\mathcal{O}_S) \simeq \mathcal{H}^i\,\mathrm{FL}(h^e_+\mathcal{O}_T) = 0 \quad \text{for} \quad i \neq 0\,, \tag{2.0.9}$$
$$\mathcal{H}^i\mathcal{R}^\circ(g^e_\dagger\mathcal{O}_S) \simeq \mathcal{H}^i\,\mathrm{FL}(h^e_\dagger\mathcal{O}_T) = 0 \quad \text{for} \quad i \neq 0\,, \tag{2.0.10}$$

by Proposition 2.3(3), the exactness of h^e_+ resp. $h^e_\dagger$ (cf. Lemma 2.2(1)) and the exactness of the Fourier-Laplace transformation. Additionally the following holds:

$$\mathcal{R}(g^e_+\mathcal{O}_S) \simeq \varphi_{B',+}\mathcal{O}_{S\times W} \in D^{\leq 0}_{rh}(\mathcal{D}_V)\,, \tag{2.0.11}$$
$$\mathcal{R}(g^e_\dagger\mathcal{O}_S) \simeq \varphi_{B',\dagger}\mathcal{O}_{S\times W} \in D^{\geq 0}_{rh}(\mathcal{D}_V)\,, \tag{2.0.12}$$

where the isomorphisms hold by Proposition 2.3(1) and the statement about the cohomology of $\varphi_{B',+}\mathcal{O}_{S\times W}$ holds because $\varphi_{B'}$ is affine and therefore $\varphi_{B',+}$ is right exact. The claim about the cohomology of $\varphi_{B',\dagger}\mathcal{O}_{S\times W}$ follows by duality.

Now we take the long exact cohomology sequence of the two triangles in (2.0.8). For the first triangle we get

$$0 \longrightarrow \mathcal{H}^{-1}(\mathcal{R}_{cst}(g^e_+\mathcal{O}_S)) \longrightarrow \mathcal{H}^0(\mathcal{R}(g^e_+\mathcal{O}_S)) \longrightarrow \mathcal{H}^0(\mathcal{R}^\circ_c(g^e_+\mathcal{O}_S)) \longrightarrow \mathcal{H}^0(\mathcal{R}_{cst}(g^e_+\mathcal{O}_S)) \longrightarrow 0$$

and

$$\mathcal{H}^{i-1}(\mathcal{R}_{cst}(g^e_+\mathcal{O}_S)) \simeq \mathcal{H}^i(\mathcal{R}(g^e_+\mathcal{O}_S)) \quad \text{for} \quad i \leq -1\,,$$

because of (2.0.9) and (2.0.11). For the second triangle we get

$$0 \to \mathcal{H}^0(\mathcal{R}_{cst}(g^e_\dagger\mathcal{O}_S)) \to \mathcal{H}^0(\mathcal{R}^\circ(g^e_\dagger\mathcal{O}_S)) \to \mathcal{H}^0(\mathcal{R}(g^e_\dagger\mathcal{O}_S)) \to \mathcal{H}^1(\mathcal{R}_{cst}(g^e_\dagger\mathcal{O}_S)) \to 0$$

and

$$\mathcal{H}^i(\mathcal{R}(g^e_\dagger\mathcal{O}_S)) \simeq \mathcal{H}^{i+1}(\mathcal{R}_{cst}(g^e_\dagger\mathcal{O}_S)) \quad \text{for} \quad i \geq 1\,,$$

because of (2.0.10) and (2.0.12). Applying Proposition 2.3 to the single terms shows the claims. $\square$

In order to relate $\mathrm{FL}(h^e_+\mathcal{O}_T)$ resp. $\mathrm{FL}(h^e_\dagger\mathcal{O}_T)$ to GKZ-systems, we need the following lemma.

Lemma 2.5. *There is the following isomorphism in $D^b_{rh}(\mathcal{D}_V)$:*

$$\mathrm{FL}(h^e_+\mathcal{O}_T) \simeq \bigoplus_{\gamma \in I_e} \mathrm{FL}(h_+\mathcal{O}_T\underline{y}^{(0,\gamma)})\,,$$

where I_e is the set $\prod_{k=1}^d \frac{([0,e_k-1]\cap\mathbb{N})}{e_k} \subset \mathbb{Q}^d$.

Proof. Recall that we have $h^e = h \circ \widetilde{\pi}^e$ with

$$\begin{aligned} \widetilde{\pi}^e : T &\longrightarrow T\,, \\ (y_0, y_1, \ldots, y_d) &\mapsto (y_0, y_1^{e_1}, \ldots, y_d^{e_d})\,, \end{aligned} \tag{2.0.13}$$

so we just have to compute $\widetilde{\pi}^e_+ \mathcal{O}_T$. A simple computation shows that

$$\begin{aligned} \widetilde{\pi}^e_+ \mathcal{O}_T &\simeq \bigoplus_{\gamma \in I_e} \mathcal{D}_T / (\mathcal{D}_T y_0 \partial_{y_0} + \mathcal{D}_T (y_1 \partial_{y_1} - \gamma_1) + \ldots + \mathcal{D}_T (y_d \partial_{y_d} - \gamma_d)) \\ &= \bigoplus_{\gamma \in I_e} \mathcal{O}_T \underline{y}^{(0,\gamma)}\,. \end{aligned}$$

□

As above we denote by B be a $d \times n$ integer matrix, whose columns generate $\mathbb{Q}^d$. Using the Smith normal form, the matrix B can be written as

$$B = C \cdot D_1 \cdot D_2 \cdot M = C \cdot \begin{pmatrix} e_1 & & \\ & \ddots & \\ & & e_d \end{pmatrix} \cdot \left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \end{array} \right) \cdot M$$

where $D \in Gl(d \times d, \mathbb{Z})$, $M \in Gl(n \times n, \mathbb{Z})$, $B' = D_1 \cdot D_2 \cdot M$ and $A = D_2 \cdot M$.

Theorem 2.6. *Let B and $\widetilde{A}$ be as above and let $\varphi_B : S \times W \longrightarrow \mathbb{C}_{\lambda_0} \times W$ be the corresponding Laurent polynomial. There exist a $\widetilde{\delta}, \widetilde{\delta}' \in \mathbb{N}\widetilde{A}$ such that for every $\widetilde{\beta} \in \widetilde{\delta} + (\mathbb{Q}_+ \widetilde{A} \cap \mathbb{Z}^{d+1})$ resp. $\widetilde{\beta}' \in \widetilde{\delta}' + (\mathbb{Q}_+ \widetilde{A} \cap \mathbb{Z}^{d+1})$ we have the following exact sequences in $M_{rh}(\mathcal{D}_V)$:*

$$0 \longrightarrow H^{d-1}(S, \mathbb{C}) \otimes \mathcal{O}_V \longrightarrow \mathcal{H}^0(\varphi_{B,+} \mathcal{O}_{S \times W}) \longrightarrow \bigoplus_{\gamma \in I_e} \mathcal{M}_{\widetilde{A}}^{\widetilde{\beta} + (0,\gamma)} \longrightarrow H^d(S, \mathbb{C}) \otimes \mathcal{O}_V \longrightarrow 0\,,$$

$$0 \longrightarrow H_d(S, \mathbb{C}) \otimes \mathcal{O}_V \longrightarrow \bigoplus_{\gamma \in I_e} \mathcal{M}_{\widetilde{A}}^{-\widetilde{\beta}' - (0,\gamma)} \longrightarrow \mathcal{H}^0(\varphi_{B,\dagger} \mathcal{O}_{S \times W}) \longrightarrow H_{d-1}(S, \mathbb{C}) \otimes \mathcal{O}_V \longrightarrow 0\,.$$

Proof. By Proposition 2.4 the only thing which we have to show for the first sequence, is the identification of $\mathrm{FL}(h^e_+ \mathcal{O}_T)$ with $\bigoplus_{\gamma \in I_e} \mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}+(0,\gamma)}$. Notice that we have

$$\mathrm{FL}(h^e_+ \mathcal{O}_T) \simeq \bigoplus_{\gamma \in I_e} \mathrm{FL}(h_+ \mathcal{O}_T \underline{y}^{(0,\gamma)}) \simeq \bigoplus_{\gamma \in I_e} \mathrm{FL}(h_+ \mathcal{O}_T \underline{y}^{\widetilde{\alpha} + (0,\gamma)})$$

by Lemma 2.5 and the fact that $\mathcal{O}_T \underline{y}^{(0,\gamma)} \simeq \mathcal{O}_T \underline{y}^{\widetilde{\alpha}+(0,\gamma)}$ for every $\widetilde{\alpha} \in \mathbb{Z}^{d+1}$. In view of Theorem 1.15 we need to find a $\widetilde{\delta} \in \mathbb{N}\widetilde{A}$, so that for every $\widetilde{\beta} = (\beta_0, \beta) \in \widetilde{\delta} + (\mathbb{Q}_+ \widetilde{A} \cap \mathbb{Z}^{d+1})$ the set $\{\widetilde{\beta} + (0,\gamma) \in \mathbb{Q}^d \mid \gamma \in I_e\}$ has empty intersection with $sRes(\widetilde{A})$. Lemma 1.16 shows that there exists an $\delta_{\widetilde{A}} \in \mathbb{N}\widetilde{A}$ so that $(\delta_{\widetilde{A}} + \mathbb{R}_+ \widetilde{A}) \cap sRes(\widetilde{A}) = \emptyset$. But because $\mathbb{R}_+ \widetilde{A}$ is a cone with non-empty interior we surely can find an $\widetilde{\delta} \in \delta_{\widetilde{A}} + \mathbb{N}\widetilde{A}$ so that for every $\widetilde{\beta} \in \widetilde{\delta} + (\mathbb{Q}_+ \widetilde{A} \cap \mathbb{Z}^{d+1})$ the set above has empty intersection with $sRes(\widetilde{A})$. This shows the existence of the first exact sequence.
By Proposition 1.23 the holonomic dual of $\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}+(0,\gamma)}$ is isomorphic to $\mathcal{M}_{\widetilde{A}}^{-\widetilde{\beta}-(0,\gamma)+\kappa}$ for all $\kappa \in \mathbb{Z}^{d+1}$ such that $-\widetilde{\beta} - (0,\gamma) + \kappa \in -(\mathbb{R}_+ \widetilde{A})^\circ$. But we have $\widetilde{\beta} + (0,\gamma) \in \widetilde{\delta} + \mathbb{R}_+ \widetilde{A} \subset \mathbb{R}_+ \widetilde{A}$ and therefore $-\widetilde{\beta} - (0,\gamma) \in -\mathbb{R}_+ \widetilde{A}$. Choose some $\rho \in (\mathbb{R}_+ \widetilde{A})^\circ \cap \mathbb{Z}^{d+1}$, then $-\widetilde{\beta} - \rho - (0,\gamma) \in -\rho - \mathbb{R}_+ \widetilde{A} \subset -(\mathbb{R}_+ \widetilde{A})^\circ$. Now set $\widetilde{\delta}' := \widetilde{\delta} + \rho$ and choose some $\widetilde{\beta}' \in \widetilde{\delta}' + (\mathbb{Q}_+ \widetilde{A} \cap \mathbb{Z}^{d+1})$. Then $\widetilde{\beta} := \widetilde{\beta}' - \rho \in \widetilde{\delta} + (\mathbb{Q}_+ \widetilde{A} \cap \mathbb{Z}^{d+1})$ and therefore $\mathbb{D}\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}+(0,\gamma)} \simeq \mathcal{M}_{\widetilde{A}}^{-\widetilde{\beta}'-(0,\gamma)}$. This shows the claim. □

In the case where the columns of B generate $\mathbb{Z}^d$, i.e. in the case $B = A$, we can be more precise with respect to the allowed parameter vector β.

Theorem 2.7. *Let A be an integer $d \times n$-matrix with $\mathbb{Z}A = \mathbb{Z}^d$ and let φ_A be the corresponding family Laurent polynomials. For every $\widetilde{\beta}' \in (\mathbb{R}_+\widetilde{A})^\circ \cap \mathbb{Z}^{d+1}$ we have the following exact sequence in $M_{rh}(\mathcal{D}_V)$:*

$$0 \longrightarrow H_d(S,\mathbb{C}) \otimes \mathcal{O}_V \longrightarrow \mathcal{M}_{\widetilde{A}}^{-\widetilde{\beta}'} \longrightarrow \mathcal{H}^0(\varphi_{A,\dagger}\mathcal{O}_{S\times W}) \longrightarrow H_{d-1}(S,\mathbb{C}) \otimes \mathcal{O}_V \longrightarrow 0\,. \qquad (2.0.14)$$

For every $\widetilde{\beta} \in \mathbb{Z}^{d+1}$ with $\widetilde{\beta} \notin sRes(\widetilde{A})$ we have the following exact sequence:

$$0 \longrightarrow H^{d-1}(S,\mathbb{C}) \otimes \mathcal{O}_V \longrightarrow \mathcal{H}^0(\varphi_{A,+}\mathcal{O}_{S\times W}) \longrightarrow \mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}} \longrightarrow H^d(S,\mathbb{C}) \otimes \mathcal{O}_V \longrightarrow 0\,. \qquad (2.0.15)$$

If in addition $\mathbb{N}\widetilde{A}$ is saturated, the set $\{\widetilde{\beta} \in \mathbb{Z}^{d+1} \mid \widetilde{\beta} \notin sRes(\widetilde{A}\}$ is precisely $\mathbb{N}\widetilde{A}$.

Proof. First notice that in the case $B = B' = A$ we have $h^e = h$. By Proposition 2.4, we have to show $\mathrm{FL}(h_+\mathcal{O}_T) \simeq \mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}}$ for every $\widetilde{\beta} \in \mathbb{Z}^{d+1}$ with $\widetilde{\beta} \notin sRes(\widetilde{A})$. But this follows from Theorem 1.15 and the fact that $\mathcal{O}_T \simeq \mathcal{O}_T^{\widetilde{\alpha}}$ for every $\widetilde{\alpha} \in \mathbb{Z}^{d+1}$.

If $\mathbb{N}\widetilde{A}$ is saturated we have $\mathbb{N}\widetilde{A} \subset \mathbb{Z}^{d+1} \setminus sRes(\widetilde{A})$ by Lemma 1.17. Now let $\widetilde{\beta} \in \mathbb{N}\widetilde{A}$ and $\widetilde{\beta}' \in \mathbb{Z}^{d+1} \setminus sRes(\widetilde{A})$, then

$$\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}} \simeq \mathrm{FL}(h_+\mathcal{O}_T) \simeq \mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}'}\,.$$

Using the classification of Theorem 1.21 above, we see that $F_\sigma(\widetilde{\beta}') = F_\sigma(\widetilde{\beta}) \in \mathbb{N}$ for all facets of $\mathbb{Q}_+\widetilde{A}$. But this shows $\widetilde{\beta}' \in \mathbb{N}\widetilde{A}$.

For the first exact sequence choose some $\widetilde{\alpha} = (\alpha_0, \alpha) \in \delta_{\widetilde{A}} + C_{\widetilde{A}}^{\mathbb{Z}}$ (cf. Lemma 1.16). By the proof of the first part we get the following exact sequence

$$0 \longrightarrow H^{d-1}(S,\mathbb{C}) \otimes \mathcal{O}_V \longrightarrow \mathcal{H}^0(\varphi_{A,+}\mathcal{O}_{S\times W}) \longrightarrow \mathcal{M}_{\widetilde{A}}^{\widetilde{\alpha}} \longrightarrow H^d(S,\mathbb{C}) \otimes \mathcal{O}_V \longrightarrow 0 \qquad (2.0.16)$$

By Proposition 1.23 the holonomic dual of $\mathcal{M}_{\widetilde{A}}^{\widetilde{\alpha}}$ is isomorphic to $\mathcal{M}_{\widetilde{A}}^{-\widetilde{\alpha}+\kappa}$ for all $\kappa \in \mathbb{Z}^{d+1}$ with $-\widetilde{\alpha} + \kappa \in -(\mathbb{R}_+\widetilde{A})^\circ \cap \mathbb{Z}^{d+1}$. The first exact sequence follows now from dualizing (2.0.16).

□

If the semigroup $\mathbb{N}\widetilde{A}$ is saturated and $\mathbb{N}A = \mathbb{Z}^d$ we can compute the morphism between the Gauß-Manin system $\mathcal{H}^0(\varphi_+\mathcal{O}_{S\times W})$ and the GKZ system $\mathcal{M}_{\widetilde{A}}^0$ as well as the kernel and cokernel. We will first deduce the description of the Gauß-Manin system by relative differential forms. This description is well-known to the experts but the author could not find a suitable reference. To compute $\mathcal{H}^0(\varphi_+\mathcal{O}_{S\times W})$ we factor the map φ into a closed embedding

$$\widetilde{i} : S \times W \longrightarrow S \times V\,,$$
$$(\underline{y}, \underline{\lambda}) \mapsto (\underline{y}\,,\, F(\underline{y}, \underline{\lambda})\,,\, \underline{\lambda}) = (\underline{y}, -\sum_{i=1}^n \lambda_i \underline{y}^{\underline{a}_i}, \underline{\lambda})$$

and the projection $p : S \times V \longrightarrow V$. The image of $\widetilde{i}$ is the smooth hypersurface Γ given by $\gamma := \lambda_0 + \sum_{i=1}^n \lambda_i \underline{y}^{\underline{a}_i} = 0$.
The direct image $\mathcal{B}_{\Gamma|S\times V} := \widetilde{i}_+\mathcal{O}_{S\times W}$ is isomorphic to $\mathcal{D}_{S\times V}/I$, where the ideal I is given by

$$I = \left(\mathcal{D}_{S\times V}(\partial_{\lambda_i} - \underline{y}^{\underline{a}_i}\partial_{\lambda_0})_{i=1,\ldots,n} + \mathcal{D}_{S\times V}(\partial_{y_k} - y_k^{-1}\sum_{i=1}^n a_{ki}\lambda_i \underline{y}^{\underline{a}_i}\partial_{\lambda_0})_{k=1,\ldots,d} + \mathcal{D}_{S\times V}(\lambda_0 + \sum_{i=1}^n \lambda_i \underline{y}^{\underline{a}_i})\right).$$

Thus we can identify $\mathcal{B}_{\Gamma|S\times V}$ with $\mathcal{O}_{S\times W}[\partial_{\lambda_0}]$ which has the following action of $\mathcal{D}_{S\times V}$:

$$
\begin{aligned}
&\partial_{\lambda_0}(g\otimes\partial_{\lambda_0}^l)=g\otimes\partial_{\lambda_0}^{l+1}\,,\\
&\partial_{\lambda_i}(g\otimes\partial_{\lambda_0}^l)=(\partial_{\lambda_i}g)\otimes\partial_{\lambda_0}^l-g(\partial_{\lambda_i}F)\otimes\partial_{\lambda_0}^{l+1}=(\partial_{\lambda_i}g)\otimes\partial_{\lambda_0}^l+g\cdot\underline{y}^{\underline{a}_i}\otimes\partial_{\lambda_0}^{l+1}\,,\\
&\partial_{y_k}(g\otimes\partial_{\lambda_0}^l)=(\partial_{y_k}g)\otimes\partial_{\lambda_0}^l-g\cdot(\partial_{y_k}F)\otimes\partial_{\lambda_0}^{l+1}=(\partial_{y_k}g)\otimes\partial_{\lambda_0}^l+g\cdot y_k^{-1}(\sum_{i=1}^n a_{ki}\lambda_i\underline{y}^{\underline{a}_i})\otimes\partial_{\lambda_0}^{l+1}\,,\\
&\lambda_0(g\otimes\partial_{\lambda_0}^l)=g\cdot F\otimes\partial_{\lambda_0}^l-l\cdot g\otimes\partial_{\lambda_0}^{l-1}=g\cdot(-\sum_{i=1}^n\lambda_i\underline{y}^{\underline{a}_i})\otimes\partial_{\lambda_0}^l-l\cdot g\otimes\partial_{\lambda_0}^{l-1}\,,\\
&\lambda_i(g\otimes\partial_{\lambda_0}^l)=\lambda_i g\otimes\partial_{\lambda_0}^l\,,\\
&y_k(g\otimes\partial_{\lambda_0}^l)=y_k g\otimes\partial_{\lambda_0}^l\,.
\end{aligned}
$$

The direct image under the projection $p: S\times V\longrightarrow V$ is equal to

$$p_+\mathcal{B}_{\Gamma|S\times V}\simeq Rp_*\Omega^{\bullet+d}_{S\times V/V}(\mathcal{B}_{\Gamma|S\times V})\simeq Rp_*\left(\Omega^{\bullet+d}_S\otimes_{\mathcal{O}_S}\mathcal{B}_{\Gamma|S\times V}\right).$$

As the map p is affine, this is equal to $p_*\left(\Omega^{\bullet+d}_S\otimes_{\mathcal{O}_S}\mathcal{B}_{\Gamma|S\times V}\right)$, where the differential on the last complex is given by

$$d(\omega\otimes Q)=d\omega\otimes Q+\sum_{k=1}^d dy_k\wedge\omega\otimes\partial_{y_k}Q\,.$$

If we use the isomorphism $\mathcal{B}_{\Gamma|S\times V}\simeq\mathcal{O}_{S\times W}[\partial_{\lambda_0}]$, the latter complex becomes isomorphic to $p_*\Omega^{\bullet+d}_{S\times W/W}[\partial_{\lambda_0}]$ with differential given by

$$d(w\otimes\partial_{\lambda_0}^l)=d\omega\otimes\partial_{\lambda_0}^l-d_yF\wedge\omega\otimes\partial_{\lambda_0}^{l+1}\,.$$

Thus the Gauß-Manin system $\mathcal{H}^0(\varphi_+\mathcal{O}_{S\times W})$ is given by

$$\frac{p_*\Omega^d_{S\times W/W}[\partial_{\lambda_0}]}{(d-\partial_{\lambda_0}d_yF\wedge)p_*\Omega^{d-1}_{S\times W/W}[\partial_{\lambda_0}]}\tag{2.0.17}$$

with the following $\mathcal{D}_V$-action:

$$\partial_{\lambda_0}(\omega\otimes\partial_{\lambda_0}^l)=\omega\otimes\partial_{\lambda_0}^{l+1}\,,$$

$$\partial_{\lambda_i}(\omega\otimes\partial_{\lambda_0}^l)=(\partial_{\lambda_i}\omega)\otimes\partial_{\lambda_0}^l-\omega(\partial_{\lambda_i}F)\otimes\partial_{\lambda_0}^{l+1}=(\partial_{\lambda_i}\omega)\otimes\partial_{\lambda_0}^l+\omega\cdot\underline{y}^{\underline{a}_i}\otimes\partial_{\lambda_0}^{l+1}\,,\tag{2.0.18}$$

$$\lambda_0(\omega\otimes\partial_{\lambda_0}^l)=\omega\cdot F\otimes\partial_{\lambda_0}^l-l\cdot\omega\otimes\partial_{\lambda_0}^{l-1}=\omega\cdot(-\sum_{i=1}^n\lambda_i\underline{y}^{\underline{a}_i})\otimes\partial_{\lambda_0}^l-l\cdot\omega\otimes\partial_{\lambda_0}^{l-1}\,,\tag{2.0.19}$$

$$\lambda_i(\omega\otimes\partial_{\lambda_0}^l)=\lambda_i\omega\otimes\partial_{\lambda_0}^l$$

for $\omega\in\Omega^d_{S\times W/W}$.

If we use the element $\omega_0:=\frac{dy_1}{y_1}\wedge\ldots\wedge\frac{dy_d}{y_d}$ as a global section for the (locally) free sheaf $\Omega^d_{S\times W/W}$ of rank one, we get an isomorphism

$$\mathcal{H}^0(\varphi_+\mathcal{O}_{S\times W})\simeq p_*\left(\frac{\mathcal{O}_{S\times W}[\partial_{\lambda_0}]}{((y_k\partial_{y_k}-y_k\partial_{y_k}F\partial_{\lambda_0})(\mathcal{O}_{S\times W}[\partial_{\lambda_0}]))_{k=1,\ldots,d}}\;\omega_0\right).$$

Proposition 2.8.

1. *Up to multiplication with a non-zero constant the map ψ is given by*

$$\psi(\prod_{i=1}^n\underline{y}^{m_i\cdot\underline{a}_i}\omega_0\otimes\partial_{\lambda_0}^s)=\partial_{\lambda_0}^{s-m+1}\partial_{\lambda_1}^{m_1}\ldots\partial_{\lambda_n}^{m_n}$$

where $m=\sum_{i=1}^n m_i$.

2. The image of ψ in $\mathcal{M}_{\widetilde{A}}^{(0,\underline{0})}$ is equal to the submodule generated by $\partial_{\lambda_0}, \ldots, \partial_{\lambda_m}$.

3. The kernel $\mathbb{V}^{n-1}$ of $\psi : \mathcal{H}^0(\widetilde{\varphi}_+ \mathcal{O}_{S_0 \times W}) \longrightarrow \mathcal{M}_{\widetilde{A}}^{(0,\underline{0})}$ is spanned by n flat sections given by

$$\sum_{i=1}^{m} a_{ki} \lambda_i \underline{y}^{\underline{a}_i} \cdot \omega_0 \qquad \text{for} \quad k = 1, \ldots, n\,.$$

Proof.
In the course of the proof we will use the modules of global sections instead of the $\mathcal{D}$-modules themselves.

First, we prove that $\psi(\omega_0) \neq 0$. As ψ is not equal zero and D_V-linear (in particular $\mathcal{O}_V$-linear), there is an element $b = \prod_{i=1}^n \underline{y}^{m_i \cdot \underline{a}_i} \omega_0 \otimes \partial_{\lambda_0}^s$ with $m_i \in \mathbb{Z}$ for $i = 1, \ldots, n$ such that $\psi(b) \neq 0$. Recall that we have $\partial_{\lambda_0}^{n_0} \cdot \prod_{i=1}^n \partial_{\lambda_i}^{n_i} \cdot \omega_0 = \prod_{i=1}^n \underline{y}^{n_i \cdot \underline{a}_i} \omega_0 \otimes \partial_{\lambda_0}^{\widetilde{n}}$ for $\widetilde{n} = \sum_{i=0}^n n_i$ and $n_i \in \mathbb{N}$. Let $I = \{i_1, \ldots, i_r\} = \{i \mid m_i < 0\}$, $I^c := \{1, \ldots n\} \setminus I$ and set $m_I := \sum_{i \in I} (-m_i)$. We have $\psi(\prod_{i \in I^c} \underline{y}^{m_i \underline{a}_i} \omega_0 \otimes \partial_{\lambda_0}^{s+m_I+k}) = \partial_{\lambda_0}^k \prod_{i \in I} \partial_{\lambda_i}^{-m_i} \psi(b)$ for every $k \geq 0$. Notice that $\psi(\prod_{i \in I^c} \underline{y}^{m_i \underline{a}_i} \omega_0 \otimes \partial_{\lambda_0}^{s+m_I+k}) \neq 0$ because for every $j \in \{0, \ldots, n\}$ the element ∂_{λ_j} acts bijectively on $M_{\widetilde{A}}^{(0,\underline{0})}$ (cf. Theorem 1.15(3)). Set $k = \max\{0, \sum_{i \in I^c} m_i - s - m_I\} = \max\{0, m - s\}$. The element $\prod_{i \in I^c} \underline{y}^{m_i \underline{a}_i} \omega_0 \otimes \partial_{\lambda_0}^{s+m_I+k}$ can be written as $P \cdot \omega_0$ for $P = \partial_{\lambda_0}^{s-m+k} \prod_{i \in I^c} \partial_{\lambda_i}^{m_i} \in \mathcal{D}_V$. We conclude that $0 \neq \psi(P \cdot \omega_0) = P \cdot \psi(\omega_0)$, which shows $\psi(\omega_0) \neq 0$.

The element ω_0 satisfies the following relations:

$$\begin{aligned}
(\lambda_0 \partial_{\lambda_0} + \sum_{i=1}^n \lambda_i \partial_{\lambda_i}) \cdot \omega_0 = -\omega_0\,, \\
\sum_{i=1}^n a_{ki} \lambda_i \partial_{\lambda_i} \cdot \omega_0 = \sum_{i=1}^n a_{ki} \lambda_i \underline{y}^{\underline{a}_i} \omega_0 \otimes \partial_{\lambda_0} = y_k \partial_{y_k} \cdot \omega_0 = 0\,, \\
\Box_{\underline{l}} \cdot \omega_0 = \left(\prod_{i: l_i < 0} \underline{y}^{l_i \cdot \underline{a}_i} - \prod_{i: l_i > 0} \underline{y}^{l_i \cdot \underline{a}_i} \right) \omega_0 \otimes \partial_{\lambda_0}^l = 0
\end{aligned}$$

for $l = \deg(\Box_{\underline{l}})$. This shows the existence of a morphism $M_{\widetilde{A}}^{-1,\underline{0}} \longrightarrow \Gamma(V, \mathcal{H}^0(\varphi_+ \mathcal{O}_{S \times W}))$ which sends 1 to ω_0. If we concatenate this morphism with the morphism

$$\psi : \Gamma(V, \mathcal{H}^0(\widetilde{\varphi}_+ \mathcal{O}_{S_0 \times W})) \longrightarrow M_{\widetilde{A}}^{0,\underline{0}}\,,$$

we get a non-zero morphism $M_{\widetilde{A}}^{-1,\underline{0}} \longrightarrow M_{\widetilde{A}}^{0,\underline{0}}$. Now the only non-zero D_V-linear morphism (up to a constant) from $M_{\widetilde{A}}^{-1,\underline{0}}$ to $M_{\widetilde{A}}^{0,\underline{0}}$ is right multiplication with ∂_{λ_0} (cf. Proposition 1.24). But this shows that the image of ω_0 in $M_{\widetilde{A}}^{0,\underline{0}}$ is ∂_{λ_0} (after a possible multiplication with a non-zero constant).

From the discussion above we get now for some general $b = \prod_{i=1}^n \underline{y}^{m_i \cdot \underline{a}_i} \omega_0 \otimes \partial_{\lambda_0}^s$ the following identities

$$\partial_{\lambda_0}^k \prod_{i \in I} \partial_{\lambda_i}^{-m_i} \psi(b) = \psi(\prod_{i \in I^c} \underline{y}^{m_i \underline{a}_i} \omega_0 \otimes \partial_{\lambda_0}^{s+m_I+k}) = \partial_{\lambda_0}^{s-m+k} \prod_{i \in I^c} \partial_{\lambda_i}^{m_i} \psi(\omega_0)$$

Because left multiplication with respect to all ∂_{λ_j} is bijective in $M_{\widetilde{A}}^{0,\underline{0}}$ (cf. Theorem 1.15(3)), this gives

$$\psi(b) = \partial_{\lambda_0}^{s-m} \prod_{i=1}^n \partial_{\lambda_i}^{m_i} \psi(\omega_0)\,.$$

This shows the first point.

In particular $\partial_{\lambda_1}, \dots, \partial_{\lambda_n}$ is in the image of the map $\Gamma(V, \mathcal{H}^0(\widetilde{\varphi}_+ \mathcal{O}_{S\times W})) \longrightarrow M^{\underline{0},\underline{0}}_{\widetilde{A}}$. We conclude that the submodule of $M^{\underline{0},\underline{0}}_{\widetilde{A}}$ which is generated by $\partial_{\lambda_0}, \dots, \partial_{\lambda_n}$ lies in the image. Notice that 1 does not lie in the image because otherwise the map would be surjective. But this shows that the image is in fact equal to the submodule generated by $\partial_{\lambda_0}, \dots, \partial_{\lambda_n}$ as the cokernel $H^d(S, \mathbb{C}) \otimes \mathcal{O}_V$ has no $\mathcal{O}_V$-torsion. This shows the second point.

Consider the elements

$$f_k := \sum_{i=1}^n a_{ki} \lambda_i \underline{y}^{\underline{a}_i} \omega_0 \qquad \text{for} \quad k = 1, \dots, d\,.$$

Their image in $M^{(\underline{0},\underline{0})}_{\widetilde{A}}$ is equal to $\sum_{i=1}^n a_{ki}\lambda_i \partial_{\lambda_i}$ which in turn is equal to 0. Thus the f_k lie in the kernel of ψ. It remains to show that they are flat:

$$\begin{aligned}
\partial_{\lambda_0} \cdot \sum_{i=1}^n a_{ki}\lambda_i \underline{y}^{\underline{a}_i}\omega_0 &= \sum_{i=1}^n a_{ki}\lambda_i \underline{y}^{\underline{a}_i}\omega_0 \otimes \partial_{\lambda_0} = y_k \partial_{y_k} \cdot \omega_0 = 0\,,\\
\partial_{\lambda_l} \cdot \sum_{i=1}^n a_{ki}\lambda_i \underline{y}^{\underline{a}_i}\omega_0 &= a_{kl}\underline{y}^{\underline{a}_l}\omega_0 + \left(\sum_{i=1}^n a_{ki}\lambda_i \underline{y}^{\underline{a}_i}\right) \underline{y}^{\underline{a}_l}\omega_0 \otimes \partial_{\lambda_0} = y_k \partial_{y_k} \cdot \underline{y}^{\underline{a}_l}\omega_0 = 0\,.
\end{aligned}$$

This shows the third point. □

Remark 2.9. *Notice that the first formula in Proposition 2.8 might involve negative powers of ∂_{λ_j}. By Theorem 1.15 3. this is well-defined, i.e. the element $\partial_{\lambda_0}^{m-s+1}\partial_{\lambda_1}^{m_1}\dots\partial_{\lambda_n}^{m_n}$ is the unique element $P \in M^{\underline{0},\underline{0}}_{\widetilde{A}}$ so that for $k = \max\{0, m-s+1\}$ we have $\partial_{\lambda_0}^{k-m+s-1}\prod_{i\in I}\partial_{\lambda_i}^{-m_i} \cdot P = \partial_{\lambda_0}^k \prod_{i\in I^c}\partial_{\lambda_i}^{m_i}$. Computing this element P in general seems to be difficult. Consider the GKZ system $M^{0,0}_{\widetilde{A}}$ with*

$$\widetilde{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

A straightforward computation shows that the element $\partial_{\lambda_0}^{-1}$ in $M^{0,0}_{\widetilde{A}}$ is equal to $(\lambda_0^2 - 4\lambda_1\lambda_2)\partial_{\lambda_0} + \lambda_0$. One can see in this example that the expression involves the discriminant of the associated family of Laurent polynomials ($A = (1, -1)$):

$$\begin{aligned}
\varphi_A : \mathbb{C}^* \times \mathbb{C}^2 &\longrightarrow \mathbb{C}_{\lambda_0} \times \mathbb{C}^2\,,\\
(y, \lambda_1, \lambda_2) &\mapsto (-\lambda_1 y - \lambda_2 \frac{1}{y}, \lambda_1, \lambda_2)\,.
\end{aligned} \tag{2.0.20}$$

3 Hypergeometric systems and Mixed Hodge Modules

In this section we show that we can endow a GKZ hypergeometric system with integer parameter with a structure of a mixed Hodge module in the sens of [Sai90]. First we show that the exact sequences in Theorem 2.6 and 2.7 are actually exact sequences in the category of mixed Hodge modules.

With regard to these theorems this might be expected as the other three terms of the exact sequences carry a natural structure of a mixed Hodge module (the two outer terms are actually (constant) variations of mixed Hodge structures). However we can not conclude directly that the (direct sum of) GKZ systems carry a mixed Hodge module structure because the category of mixed Hodge modules is not stable by extension.

Notice that the various functors $\mathcal{R}, \mathcal{R}_{cst}, \mathcal{R}^\circ_{(c)}$ are just a concatenation of (proper) direct image functors and (exceptional) inverse image functors, which means they are also defined in the derived category of (algebraic) mixed Hodge modules.

Proposition 3.1. *Let B, $\widetilde{A}$, $\widetilde{\delta}$ and $\widetilde{\delta}'$ be as in Theorem 2.6. For every $\widetilde{\beta} \in \widetilde{\delta} + (\mathbb{Q}_+ \widetilde{A} \cap \mathbb{Z}^{d+1})$ resp. $\widetilde{\beta}' \in \widetilde{\delta}' + (\mathbb{Q}_+ \widetilde{A} \cap \mathbb{Z}^{d+1})$ we have the following exact sequences in $MHM(V)$:*

$$0 \to \mathcal{H}^{n+1}(H^{d-1}(S,\mathbb{C}) \otimes \mathbb{Q}^H_V) \to \mathcal{H}^{d+n}(\varphi_{B,*}\mathbb{Q}_{S\times W}) \to \bigoplus_{\gamma \in I_e} \mathcal{M}^{\widetilde{\beta}+(0,\gamma)}_{\widetilde{A}} \to \mathcal{H}^{n+1}(H^d(S,\mathbb{C}) \otimes \mathbb{Q}^H_V) \to 0\,,$$

$$0 \to \mathcal{H}^{-n-1}(H_d(S,\mathbb{C}) \otimes \mathbb{D}\mathbb{Q}^H_V) \to \bigoplus_{\gamma \in I_e} \mathcal{M}^{-\widetilde{\beta}'-(0,\gamma)}_{\widetilde{A}} \to \mathcal{H}^{-d-n}(\varphi_{B,!}\mathbb{D}\mathbb{Q}^H_{S\times W}) \to \mathcal{H}^{-n-1}(H_{d-1}(S,\mathbb{C}) \otimes \mathbb{D}\mathbb{Q}^H_V) \to 0\,.$$

Proof. Recall that we derived the first exact sequence of Theorem 2.6 by taking the long exact cohomology sequence of the second triangle in (2.0.8):

$$\begin{tikzcd}
H^{d-1}(S,\mathbb{C}) \otimes \mathcal{O}_V \arrow[r] & \mathcal{H}^0(\varphi_{B,+}\mathcal{O}_{S\times W}) \arrow[r] & \bigoplus_{\gamma\in I_e} \mathcal{M}^{\widetilde{\beta}+(0,\gamma)}_{\widetilde{A}} \arrow[r] & H^d(S,\mathbb{C}) \otimes \mathcal{O}_V \\
\mathcal{H}^{-1}(\mathcal{R}_{cst}(g^e_+\mathcal{O}_S)) \arrow[r] \arrow[u, "\simeq"] & \mathcal{H}^0(\mathcal{R}(g^e_+\mathcal{O}_S)) \arrow[r] \arrow[u, "\simeq"] & \mathcal{H}^0(\mathcal{R}^\circ_c(g^e_+\mathcal{O}_S)) \arrow[r] \arrow[u, "\simeq"] & \mathcal{H}^0(\mathcal{R}_{cst}(g^e_+\mathcal{O}_S)) \arrow[u, "\simeq"]
\end{tikzcd}$$

where we used Proposition 2.3 1., 2. for the first, second and fourth isomorphism. For the third isomorphism we used the following isomorphisms

$$\bigoplus_{\gamma \in I_e} \mathcal{M}^{\widetilde{\beta}+(0,\gamma)}_{\widetilde{A}} \simeq \mathrm{FL}(h^e_+\mathcal{O}_T) \simeq \mathcal{R}^\circ_c(g^e_+\mathcal{O}_S)\,,$$

To show that the lower sequence is a sequence of mixed Hodge modules we replace the $\mathcal{D}$-module $\mathcal{O}_S$ with the mixed Hodge module $\mathbb{Q}^H_S := a^*_S\mathbb{Q}^H_{pt}$ and apply the corresponding functors in the (derived) category of mixed Hodge modules. Notice that there is a subtle point. In Saito's theory $\mathbb{Q}^H_X$ lies in degree $\dim X$ and for $f: X \to Y$ the functors $f^*, f^!$ correspond to the functors $f^+[-\dim X + \dim Y]$ resp. $f^\dagger[\dim X - \dim Y]$ on the level of $\mathcal{D}$-modules. If we translate the proofs above into the category of mixed Hodge modules and take these shifts into account we get

$$\begin{tikzcd}
\mathcal{H}^{n+1}(H^{d-1}(S,\mathbb{C}) \otimes \mathbb{Q}^H_V) \arrow[r] & \mathcal{H}^{d+n}(\varphi_{B,*}\mathbb{Q}^H_{S\times W}) \arrow[r] & \bigoplus_{\gamma\in I_e} \mathcal{M}^{\widetilde{\beta}+(0,\gamma)}_{\widetilde{A}} \arrow[r] & \mathcal{H}^{n+1}(H^d(S,\mathbb{C}) \otimes \mathbb{Q}^H_V) \\
\mathcal{H}^{d+n}({}^*\mathcal{R}_{cst}(g^e_*\mathbb{Q}^H_S)) \arrow[r] \arrow[u, "\simeq"] & \mathcal{H}^{d+n}({}^*\mathcal{R}(g^e_*\mathbb{Q}^H_S)) \arrow[r] \arrow[u, "\simeq"] & \mathcal{H}^{d+n+1}({}^*\mathcal{R}^\circ_c(g^e_*\mathbb{Q}^H_S)) \arrow[r] \arrow[u, "\simeq"] & \mathcal{H}^{d+n+1}({}^*\mathcal{R}_{cst}(g^e_*\mathbb{Q}^H_S)) \arrow[u, "\simeq"]
\end{tikzcd}$$

The lower sequence is an exact sequence of mixed Hodge modules by construction. If we induce the mixed Hodge module structure of $\mathcal{H}^{d+n+1}({}^*\mathcal{R}^\circ_c(g^e_*\mathbb{Q}^H_S))$ on $\bigoplus_{\gamma\in I_e} \mathcal{M}^{\widetilde{\beta}+(0,\gamma)}$ the upper sequence becomes a sequence of mixed Hodge modules, too. The statement for the second sequence follows if we dualize the two sequences above:

$$\begin{tikzcd}
\mathcal{H}^{-n-1}(H_d(S,\mathbb{C}) \otimes \mathbb{D}\mathbb{Q}^H_V) \arrow[r] & \bigoplus_{\gamma\in I_e} \mathcal{M}^{-\widetilde{\beta}'-(0,\gamma)}_{\widetilde{A}} \arrow[r] & \mathcal{H}^{-d-n}(\varphi_{B,!}\mathbb{D}\mathbb{Q}^H_{S\times W}) \arrow[r] & \mathcal{H}^{-n-1}(H_{d-1}(S,\mathbb{C}) \otimes \mathbb{D}\mathbb{Q}^H_V) \\
\mathcal{H}^{-d-n-1}({}^!\mathcal{R}_{cst}(g^e_!\mathbb{D}\mathbb{Q}^H_S)) \arrow[r] \arrow[u, "\simeq"] & \mathcal{H}^{-d-n-1}({}^!\mathcal{R}^\circ_c(g^e_!\mathbb{D}\mathbb{Q}^H_S)) \arrow[r] \arrow[u, "\simeq"] & \mathcal{H}^{-d-n}({}^!\mathcal{R}(g^e_!\mathbb{D}\mathbb{Q}^H_S)) \arrow[r] \arrow[u, "\simeq"] & \mathcal{H}^{-d-n}({}^!\mathcal{R}_{cst}(g^e_!\mathbb{D}\mathbb{Q}^H_S)), \arrow[u, "\simeq"]
\end{tikzcd}$$

where we have used that $\mathcal{H}^j(a_{S*}a^*_S\mathbb{Q}^H_{pt}) \simeq H^j(S,\mathbb{C})$ and $\mathcal{H}^j(a_{S!}a^!_S\mathbb{Q}^H_{pt}) \simeq H_{-j}(S,\mathbb{C})$ as an isomorphism of mixed Hodge structures. $\square$

Proposition 3.2. *Let A be an integer $d\times n$-matrix with $\mathbb{Z}A = \mathbb{Z}^d$. For every $\widetilde{\beta}' \in (\mathbb{R}_+\widetilde{A})^\circ \cap \mathbb{Z}^{d+1}$ we have the following exact sequence in $MHM(V)$:*

$$0 \to \mathcal{H}^{-n-1}(H_d(S,\mathbb{C})\otimes\mathbb{D}\mathbb{Q}^H_V) \to \mathcal{M}^{-\widetilde{\beta}'}_{\widetilde{A}} \to \mathcal{H}^{-d-n}(\varphi_{B,!}\mathbb{D}\mathbb{Q}^H_{S\times W}) \to \mathcal{H}^{-n-1}(H_{d-1}(S,\mathbb{C})\otimes\mathbb{Q}^H_V) \to 0\,.$$

For every $\widetilde{\beta} \in \mathbb{Z}^{d+1}$ with $\widetilde{\beta} \notin sRes(\widetilde{A})$ we have the following exact sequence:

$$0 \to \mathcal{H}^{n+1}(H^{d-1}(S,\mathbb{C}) \otimes \mathbb{Q}^H_V) \longrightarrow \mathcal{H}^{d+n}(\varphi_{A,*}\mathbb{Q}^H_{S\times W}) \longrightarrow \mathcal{M}^{\widetilde{\beta}}_{\widetilde{A}} \longrightarrow \mathcal{H}^{n+1}(H^d(S,\mathbb{C}) \otimes \mathbb{Q}^H_V) \to 0\,.$$

Proof. The proof of this proposition is parallel to the proof of Proposition 3.1. One just has to use Theorem 2.7 instead of Theorem 2.6. □

Up to now we could only handle the case $\widetilde{\beta} = (\beta_0,\beta)$ with $\beta_0 \in \mathbb{Z}$. Let A be a $d\times n$ matrix with upper row $(1,\dots,1)$ and let the parameter vector $\beta \notin sRes(A)$ resp. $-\beta \in (\mathbb{R}_+A)^\circ \cap \mathbb{Q}^d$. We will need the following lemma.

Lemma 3.3. *Let $i_1 : \{1\}\times W \longrightarrow V = \mathbb{C}\times W$ be the canonical inclusion. Then*

$$i_1^+\mathcal{M}^{(\beta_0,\beta)}_{\widetilde{A}} \simeq \mathcal{M}^\beta_A\,.$$

where $\widetilde{A}$ is given by (0.0.1)*.*

Proof. First notice that the restriction to $\{1\}\times W$ is non-characteristic due to Lemma 1.25 (1), thus we have

$$\mathcal{H}^0({}^+_1\mathcal{M}^{(\beta_0,\beta)}_{\widetilde{A}}) \simeq i_1^+\mathcal{M}^{(\beta_0,\beta)}_{\widetilde{A}}\,.$$

Recall the definition of the generators of the GKZ system from definition 1.12. Because the first row of A is equal to $(1,\dots,1)$ all operators $\Box_{l\in\mathbb{L}}$ where $\mathbb{L}$ is the lattice of relations of the matrix $\widetilde{A}$ are independent of ∂_{λ_0}. Notice also that all Euler vector fields except E_0 are independent of $\lambda_0\partial_{\lambda_0}$. Working with the D-module of global sections instead the actual $\mathcal{D}$-module, the inverse image can be written as

$$M^{(\beta_0,\beta)}_{\widetilde{A}}/(\lambda_0-1)M^{(\beta_0,\beta)}_{\widetilde{A}}\,.$$

As a D_W module this is isomorphic to

$$\mathbb{C}[\lambda_1,\dots,\lambda_n]\langle\partial_{\lambda_0},\dots,\partial_{\lambda_n}\rangle/I$$

where the ideal I is generated by the Euler fields $E_1,\dots,E_d$, the box operators $\Box_{l\in\mathbb{L}}$ and the operator

$$\partial_{\lambda_0} + \sum_{i=1}^n \lambda_i\partial_{\lambda_i}\,.$$

But this module is isomorphic to M^β_A, which shows the claim. □

Lemma 3.4. *The GKZ system $\mathcal{M}^\beta_A$ is isomorphic to a direct summand of a mixed Hodge module if $\beta\in\mathbb{Q}^d$ and if any of the following conditions are satisfied:*

1. *$\beta \notin sRes(A)$,*
2. *$\beta \in -(\mathbb{R}_+A)^\circ \cap \mathbb{Q}^d$.*

Proof. Assume that $\beta \notin sRes(A)$. Then by Lemma 1.19 there exists a $\beta_0 \in \mathbb{Z}$ such that $\widetilde{\beta} = (\beta_0,\beta) \notin sRes(\widetilde{A})$. Let $\mathfrak{F}$ be a fundamental domain for $sRes(\widetilde{A})$, i.e. by Lemma 1.18 we can find a $\widetilde{\beta}' = (\beta_0',\beta')$ lying in $\mathfrak{F}$ so that $\mathcal{M}^{\widetilde{\beta}}_{\widetilde{A}} \simeq \mathcal{M}^{\widetilde{\beta}'}_{\widetilde{A}}$. Let $\widetilde{\beta}' = \widetilde{\beta}'' + (0,\kappa)$ with $\widetilde{\beta}'' = (\beta_0'',\beta'') \in \mathbb{Z}^{d+1}$ and $\kappa \in [0,1)^d \cap \mathbb{Q}^d$. If $\kappa = 0$ we have $\mathcal{M}^\beta_A \simeq i_1^+\mathcal{M}^{\widetilde{\beta}}_{\widetilde{A}} \simeq i_1^+\mathcal{M}^{\widetilde{\beta}''}_{\widetilde{A}}$, the latter underlying a mixed Hodge module by Proposition 3.2. Thus we assume $\kappa \neq 0$ and let $e = (e_1,\dots,e_d) \in \mathbb{N}^d$ such that $e_i\cdot\kappa_i \in \mathbb{N}$ for all $i \in \{1,\dots,d\}$. By Proposition 3.1 we have induced a mixed Hodge

module structure on $\bigoplus_{\gamma\in I_e}\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}''+(0,\gamma)}$. The inverse image with respect to i_1 induces a mixed Hodge module structure on $\bigoplus_{\gamma\in I_e}\mathcal{M}_{\widetilde{A}}^{\beta''+\gamma}$. Because we have $\beta' = \beta'' + \kappa \in \bigcup_{\gamma\in I_{\underline{e}}}\beta''+\gamma$ and $\mathcal{M}_A^{\beta} \simeq i_1^+\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}} \simeq i_1^+\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}'} \simeq \mathcal{M}_A^{\beta'}$ this shows the first point.

Now let $\beta \in -(\mathbb{R}_+A)^\circ \cap \mathbb{Q}^d$. There exists an $\beta_0 \in \mathbb{Z}$ such that $\widetilde{\beta} = (\beta_0, \beta) \in -(\mathbb{R}_+\widetilde{A})^\circ \cap \mathbb{Q}^{d+1}$. Write $\widetilde{\beta} = \widetilde{\beta}' - (0,\kappa)$ for $\widetilde{\beta}' \in \mathbb{Z}^{d+1}$ and $\kappa \in [0,1)^d \cap \mathbb{Q}^d$. As above if $\kappa = 0$, then $\mathcal{M}_A^{\beta}$ is isomorphic to the underlying $\mathcal{D}$-module of a mixed Hodge module. So assume that $\kappa \neq 0$ and let $e = (e_1,\dots,e_d) \in \mathbb{N}^d$ such that $e_i \cdot \kappa_i \in \mathbb{N}$ for all $i \in \{1,\dots,d\}$.

Let $\widetilde{\beta}'' = (\beta_0'', \beta'') \in \widetilde{\delta}' + (\mathbb{Q}_+\widetilde{A} \cap \mathbb{Z}^{d+1})$. By Proposition 3.1 the $\mathcal{D}$-module $\bigoplus_{\gamma\in I_e}\mathcal{M}_{\widetilde{A}}^{-\widetilde{\beta}''-(0,\gamma)}$ underlies a mixed Hodge module. The inverse image with respect to i_1 induces a mixed Hodge module structure on $\bigoplus_{\gamma\in I_e}\mathcal{M}_{\widetilde{A}}^{-\beta''-\gamma}$. We have $\mathcal{M}_A^{\beta} \simeq i_1^+\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}} \simeq i_1^+\mathcal{M}_{\widetilde{A}}^{-\widetilde{\beta}''-(0,\kappa)} \simeq \mathcal{M}_A^{-\beta''-\kappa}$, where the second isomorphism follows from Lemma 1.20 (1). ☐

We are finally able to proof the main results of this section. Let A' be a $d\times n$ integer matrix with columns $\underline{a}_1',\dots,\underline{a}_n'$ so that there exists a linear function $h : \mathbb{Z}^d \to \mathbb{Z}$ satisfying $h(\underline{a}_i') = 1$ for all i. A GKZ system corresponding to this matrix is called homogeneous. Schulze and Walther have shown in [SW08] that a GKZ system is regular holonomic if and only if it is homogeneous.

Theorem 3.5. *The homogeneous GKZ system $\mathcal{M}_{A'}^{\beta'}$ carries a mixed Hodge module structure if $\beta' \in \mathbb{Z}^d$ and if one of the following conditions are satisfied*

1. $\beta' \notin sRes(A')$,
2. $\beta' \in -(\mathbb{R}_+A')^\circ \cap \mathbb{Z}^d$.

Proof. Let $l : \mathbb{Z}^d \to \mathbb{Z}^d$ be an isomorphism, L be the corresponding invertible integer matrix and $l_{\mathbb{C}} : \mathbb{C}^d \to \mathbb{C}^d$ its $\mathbb{C}$-linear extension. First notice that for $\beta := l(\beta')$ and $A := L\cdot A'$ the GKZ systems $\mathcal{M}_{A'}^{\beta'}$ and $\mathcal{M}_A^{\beta}$ are isomorphic. Furthermore we have $l_{\mathbb{C}}(sRes(A')) = sRes(A)$ because we have $l(deg(S_{A'})) = deg(S_A)$ and $l(-(\mathbb{R}_+A')^\circ \cap \mathbb{Z}^d) = -(\mathbb{R}_+A)^\circ \cap \mathbb{Z}^d$. It is easy to see but tedious to write down that for any A' there exists an l resp. L such that $L\cdot A'$ is a $d\times n$ matrix with first row equal to $(1,\dots,1)$. But for the corresponding GKZ system $\mathcal{M}_A^{\beta}$ the proof of Lemma 3.4 shows that it carries a mixed Hodge module structure. ☐

Theorem 3.6. *The homogeneous GKZ system $\mathcal{M}_{A'}^{\beta'}$ has quasi-unipotent monodromy if $\beta' \in \mathbb{Q}^d$ and if one of the following conditions are satisfied*

1. $\beta' \notin sRes(A)$,
2. $\beta' \in -(\mathbb{R}_+A')^\circ \cap \mathbb{Q}^d$.

Proof. As in the proof of Theorem 3.5 we can reduce to the case where the matrix A' of the GKZ system has $(1,\dots,1)$ as its first row. Lemma 3.4 shows then that $\mathcal{M}_{A'}^{\beta'}$ is isomorphic to a direct summand of a mixed Hodge module $\mathcal{N}$. There exists a stratification $\mathcal{S} = \{S_i\}$ such that the restriction to $S_i \setminus S_{i+1}$ is a smooth mixed Hodge module, i.e. it is a polarizable variation of mixed Hodge structures. Now it follows from standard Hodge theory that the underlying local system of this restriction has (local) quasi-unipotent monodromy in the sense of [Kas81]. But $\mathcal{M}_{A'}^{\beta'}$ is a direct summand of $Dmod(\mathcal{N})$ from which follows that $DR(\mathcal{M}_{A'}^{\beta'})$ is a direct summand in $rat(\mathcal{N}) \otimes \mathbb{C}$. But this shows the claim. ☐

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